

Fonctions de plusieurs variables réelles

diff

Exercice 1

$$x_0 \in \mathbb{R} \\ h \in \mathbb{R}$$

$$\begin{aligned} 1) \quad g(x_0 + h) &= (x_0 + h)^2 \\ &= \underbrace{x_0^2}_{g(x_0)} + \underbrace{2x_0h}_{L(h)} + \underbrace{h^2}_{\text{reste}} \end{aligned}$$

L est la différentielle $L(h) = 2x_0h$
linéaire continue.

$$|g(x_0 + h) - g(x_0) - L(h)| = |h^2| = |h| |h| \quad \text{on a bien } \lim_{h \rightarrow 0} |E(x_0, h)| = 0$$

$$2) \text{ Soit } h = (h_1, h_2) \in \mathbb{R}^2$$

$$\begin{aligned} g(x+h_1, y+h_2) &= (x+h_1)(y+h_2)^2 \\ &= (x+h_1)(y^2 + 2yh_2 + h_2^2) \\ &= \underbrace{xy^2}_{g(x)} + \underbrace{2xyh_2}_{L(h)} + \underbrace{xh_2^2}_{\text{reste}} + \underbrace{h_1y^2}_{\text{reste}} + \underbrace{2yh_1h_2 + h_1h_2^2}_{\text{reste}} \end{aligned}$$

$$L(h) = 2xyh_2 + h_1y^2 \quad \text{linéaire continue}$$

$$\begin{aligned} |g(x+h_1, y+h_2) - g(x, y) - L(h)| &= |xh_2^2 + 2yh_1h_2 + h_1h_2^2| \\ &\leq |x| \cdot |h_2|^2 + 2|y| |h_1| |h_2| + |h_1| |h_2|^2 \\ &\leq 3 \|(x, y)\| \cdot \|h\|^2 + \|h\|^3 \\ &\leq \|h\| \left[\underbrace{3 \|(x, y)\|}_{\|E(x, y), h\| \rightarrow 0} \cdot \|h\| + \|h\|^2 \right] \end{aligned}$$

$$3) \text{ Soit } h \in \mathbb{R}_2[X] \quad h(x) = a_h x^2 + b_h x + c_h$$

$$(p+h)(x) = (a+a_h)x^2 + (b+b_h)x + c+c_h$$

$$\begin{aligned} g(p+h)(x) &= 3(a+a_h)x^2 + (c+c_h) \\ &= \underbrace{3ax^2 + c}_{g(p)} + \underbrace{3a_h x^2 + c_h}_{g(h)} \\ &= g(p)(x) + g(h)(x) \end{aligned}$$

$$4) \quad H \in \mathcal{M}_d(\mathbb{R}) \times \mathcal{M}_d(\mathbb{R})$$

$$g(A+H_1, B+H_2)$$

$$H = (H_1, H_2)$$

$$g(A+H_1, B+H_2) = (A+H_1)^2 (B+H_2)$$

$$= \underbrace{A^2 B}_{g(A, B)} + \underbrace{A^2 H_1}_{\text{reste}} + \underbrace{A H_1 B}_{\text{reste}} + \underbrace{A H_1 H_2}_{\text{reste}} + \underbrace{H_1 A B}_{\text{reste}} + \underbrace{H_1 A H_2}_{\text{reste}} + \underbrace{H_1^2 B}_{\text{reste}} + \underbrace{H_1^2 H_2}_{\text{reste}}$$

$$\|g(A+H_1, B+H_2) - g(A, B) - L(H)\| \leq 3 \|(A, B)\| \|H\|^2 + \|H\|^3$$

5) Soit $h \in \mathbb{R}^d$

$$g(x+h) = \langle u(x+h), x+h \rangle$$

$$= \langle u(x+h), x \rangle + \langle u(x+h), h \rangle$$

$$= \underbrace{\langle u(x), x \rangle}_g + \underbrace{\langle u(h), x \rangle}_L + \underbrace{\langle u(x), h \rangle}_L + \underbrace{\langle u(h), h \rangle}_h$$

donc R

$$\|g(x+h) - g(x) - \langle h \rangle\| = |\langle u(h), h \rangle|$$

$$\leq \|u(h)\| \cdot \|h\|$$

$$\|u(h)\| \leq \|u\| \cdot \|h\|$$

$$\leq \|u\|_{\mathbb{R}^d \rightarrow \mathbb{R}} \cdot \|h\|^2$$

6) (P_h, h)

$$(P, x) + (P_h, h) = (P + P_h, x+h)$$

$$g(P + P_h, x+h) = (P + P_h)(x+h)$$

$$= P(x+h) + P_h(x+h)$$

$$= \underbrace{P(x)}_g + \underbrace{P(h)}_L + \underbrace{P_h(x)}_L + \underbrace{P_h(h)}_h$$

$$\|g(P + P_h, x+h) - g(P, x) - \langle P_h, h \rangle\|_F = \|P_h(h)\|_F$$

$$\leq \|P_h\|_{\mathcal{L}(E, F)} \|h\|_E$$

$$\leq \|(P_h, h)\|_{\mathcal{L}(E, F) \times E}^2$$

7) $h \in E$

$$g(x+h) = g(x+h)(x+h)$$

$$\text{on } g(x+h) = g(x) + \mathcal{L}_g(h) + \|h\|_E \varepsilon(x, h)$$

$$= g(x+h)(x) + g(x+h)(h)$$

$$= (g(x) + \mathcal{L}_g(h) + \|h\|_E \varepsilon(x, h))(x) + (g(x) + \mathcal{L}_g(h) + \|h\|_E \varepsilon(x, h))(h)$$

$$= \underbrace{g(x)(x)}_g + \underbrace{\mathcal{L}_g(h)(x)}_L + \underbrace{\|h\|_E \varepsilon(x, h)(x)}_L + \underbrace{g(x)(h)}_g + \underbrace{\mathcal{L}_g(h)(h)}_L + \underbrace{\|h\|_E \varepsilon(x, h)(h)}_h$$

$$\text{donc } \mathcal{L}_g(h) = \mathcal{L}_g(h)(x) + g(x)(h)$$

$$\|g(x+h) - g(x) - \mathcal{L}_g(h)\|_F = \|\|h\|_E \varepsilon(x, h)(x) + \mathcal{L}_g(h)(h) + \|h\|_E \varepsilon(x, h)(h)\|_F$$

$$+ \|\mathcal{L}_g(h)(h)\|_F$$

$$g: \mathbb{R} \rightarrow \mathcal{L}(\mathbb{R}, \mathbb{R})$$

$$x \mapsto g(x)$$

$$g(x): \mathbb{R} \rightarrow \mathbb{R}$$

$$x \mapsto \alpha x$$

$$g(x)(x) = \alpha x$$

$$\leq \|h\|_E \|\varepsilon(x, h)\|_{\mathcal{L}(E, F)} \|x\|_E + \|\mathcal{L}_g(h)\|_{\mathcal{L}(E, F)}$$

$$+ \|h\|_E \|\varepsilon(x, h)\|_{\mathcal{L}(E, F)} \|h\|_E$$

Si $y \neq 0$

$g(x, y)$ composée d'application continue donc g continue.

Si $y = 0$

$$\lim_{y \rightarrow 0} g(x, y) \stackrel{?}{=} g(x, 0)$$

$$\sin(y) = y + y^2 \varepsilon(y) \quad \text{ie} \quad \frac{\sin(y)}{y} = 1 + y \varepsilon(y)$$

$$g(x, y) = x + xy \varepsilon(y) \Rightarrow \lim_{y \rightarrow 0} g(x, y) = x \quad \text{continue en } y=0$$

En $y \neq 0$ les dérivées partielles sont:

$$\partial_1 g(x, y) = \frac{\sin y}{y}$$

$$\partial_2 g(x, y) = x \frac{\cos y \cdot y - \sin y}{y^2}$$

En $(x, 0)$

$$\frac{\partial g}{\partial x}(x, 0) = \lim_{h \rightarrow 0} \frac{g(x+h, 0) - g(x, 0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} (x+h-x) = 1$$

$$\frac{\partial g}{\partial y}(x, 0) = \lim_{h \rightarrow 0} \frac{g(x, h) - g(x, 0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} (x(\frac{h+h^2 \varepsilon(h)}{h}) - x)$$

$$= \lim_{h \rightarrow 0} x \varepsilon(h) = 0$$

$$\lim_{y \rightarrow 0} \partial_1 g(x, y) = \lim_{y \rightarrow 0} 1 + y \varepsilon(y) = 1$$

$$\sin(y) = y - \frac{y^3}{6} + y^4 \varepsilon(y)$$

$$\Rightarrow y \cos(y) - \sin(y) = -\frac{y^3}{2} + \frac{y^3}{6} + y^4 \varepsilon(y)$$

$$\cos(y) = 1 - \frac{y^2}{2} + y^3 \varepsilon(y)$$

$$= -\frac{y^3}{3} + y^4 \varepsilon(y)$$

$$y \cos(y) = y - \frac{y^3}{2} + y^4 \varepsilon(y)$$

$\rightarrow 0$

dérivées partielles continues donc C^1

Exercice 6

1) Continuité évidente

$$2) \partial_1 g_1(x, y) = 3x^2 + 1$$

$$\partial_1 g_2(x, y) = -2x$$

$$\partial_2 g_1(x, y) = 0$$

$$\partial_2 g_2(x, y) = 1$$

$$\text{Donc } J(x, y) = \begin{pmatrix} 3x^2 + 1 & 0 \\ -2x & 1 \end{pmatrix}$$

1 Les dérivées partielles sont continues donc g_1 est diff

2 Les

 g_2 est diff

donc g est diff.

Exercice 2

$$1) d(g \circ f)(a)(h) = d_1(g \circ f)(a)(h_1) + d_2(g \circ f)(a)(h_2)$$

$$g \circ f : (x, y) \xrightarrow{f} \frac{y}{x} \xrightarrow{g} g\left(\frac{y}{x}\right)$$

$$\begin{aligned} d_1 g(x, y)(h_1) &= dg\left(\frac{y}{x}\right) \circ d_1 f(x, y)(h_1) \\ &= g'\left(\frac{y}{x}\right) \left(-\frac{y}{x^2}\right) h_1 \end{aligned}$$

$$\frac{\partial g}{\partial x}(x, y) = g'\left(\frac{y}{x}\right) \left(-\frac{y}{x^2}\right)$$

$$d_2 g(x, y)(h_2) = g'\left(\frac{y}{x}\right) \left(\frac{1}{x}\right) h_2$$

$$\frac{\partial g}{\partial y}(x, y) = g'\left(\frac{y}{x}\right) \left(\frac{1}{x}\right)$$

$$2) \frac{\partial h}{\partial x}(x, y, z) = g'(z \sin(x)) z \cos(x)$$

$$\frac{\partial h}{\partial y}(x, y, z) = 0$$

$$\frac{\partial h}{\partial z}(x, y, z) = g'(z \sin(x)) \sin(x)$$

Exercice 3

$$\lim_{t \rightarrow 0} \frac{g(x+tu, y+tv) - g(x, y)}{t}$$

$$\text{em } (x, y) = (0, 0)$$

$$= \lim_{t \rightarrow 0} \frac{1}{t} \left(\frac{t^2 uv}{t^2 u^2 + t^2 v^2} \right) = 0$$

$$y = x^3 - x^2$$

$$g(x, x^3 - x^2) = \frac{x(x^3 - x^2)}{x^3} = x - 1$$

$$\lim_{x \rightarrow 0} g(x, x^3 - x^2) = -1 \quad \text{donc } g \text{ pas continue en } (0, 0)$$

Exercice 4

$$1) \forall (x, y) \neq (0, 0)$$

$$|g(x, y)| \leq \frac{|xy|}{\sqrt{x^2 + y^2}}$$

identifiez avec

$$2) \frac{\partial g}{\partial x}(x, y) = \frac{y \sqrt{x^2 + y^2} - xy \frac{1}{2} (x^2 + y^2)^{-1/2} 2x}{(x^2 + y^2)}$$

$$\leq \frac{1}{2} \sqrt{x^2 + y^2}$$

$$\rightarrow 0$$

$$x \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) + \frac{xy}{\sqrt{x^2 + y^2}} \cos\left(\frac{1}{\sqrt{x^2 + y^2}}\right)$$

$$+ \left(\frac{-\frac{1}{2} (x^2 + y^2)^{-1/2} 2x}{x^2 + y^2} \right)$$

$$= \frac{y(x^2+y^2) - yx^2}{(x^2+y^2)^{3/2}} \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{xy}{\sqrt{x^2+y^2}} \frac{x}{(x^2+y^2)^{3/2}} \cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

$$= \frac{y^3}{(x^2+y^2)^{3/2}} \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{yx^2}{(x^2+y^2)^2} \cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

$$3) \lim_{t \rightarrow 0} \frac{1}{t} \frac{g(tu, tv) - g(0,0)}{1} = \lim_{t \rightarrow 0} \frac{1}{t} \left(\frac{t^2 uv}{\sqrt{t^2 u^2 + t^2 v^2}} \sin\left(\frac{1}{t\sqrt{u^2+v^2}}\right) \right)$$

$$= \lim_{t \rightarrow 0} \frac{1}{t} \frac{uv}{\sqrt{u^2+v^2}} \sin\left(\frac{1}{t\sqrt{u^2+v^2}}\right)$$

pas de limite sauf si $u=0$ $v \neq 0$
 $u \neq 0$ $v=0$

Exercice 5

$$\frac{|x|^2 |y|^3}{x^2+y^2} \leq \frac{\|(x,y)\|^5}{\|(x,y)\|^2} \rightarrow 0$$

$$\text{car } |x| \leq \|(x,y)\|_2$$

$$|y| \leq \|(x,y)\|_2$$

$$\partial_x g(x,y) = \frac{2xy^3(x^2+y^2) - x^2 y^3(2x)}{(x^2+y^2)^2}$$

$$= \frac{2xy^5}{(x^2+y^2)^2}$$

$$\partial_z g(x,y) = \frac{3x^2 y^2 (x^2+y^2) - x^2 y^3(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^2 y^2 + 3x^2 y^2}{(x^2+y^2)^2} = \frac{x^2 y^2 (y^2 + 3x^2)}{(x^2+y^2)^2}$$

$$\lim_{(x,y) \rightarrow (0,0)} \partial_x g(x,y) = \partial_x g(0,0)$$

$$\lim_{(x,y) \rightarrow (0,0)} \partial_z g(x,y) = \partial_z g(0,0)$$

vérification

dérivée partielle au point (0,0)
selon la 1ère variable

$$\partial_x g(0,0) = \lim_{h \rightarrow 0} \frac{g(x_0+h, y_0) - g(x_0, y_0)}{h} = 0 \text{ par définition de } g.$$

$$\lim_{(x,y) \rightarrow (0,0)} \partial_x g(x,y) \leq \frac{2\|(x,y)\|^6}{\|(x,y)\|^4} = 2\|(x,y)\|^2 \rightarrow 0$$

$$\partial_z g(x,y) \leq \frac{4\|(x,y)\|^6}{\|(x,y)\|^4} = 4\|(x,y)\|^2 \rightarrow 0$$

$$\leq y^2 + 3x^2 \rightarrow 0$$

Exercice 7

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$$1) \frac{\partial g}{\partial x}(x, y, z) = 3x^2 y z$$

$$\frac{\partial g}{\partial y}(x, y, z) = x^3 z - 4yz^2$$

$$\frac{\partial g}{\partial z}(x, y, z) = x^3 y - 4yz^2$$

$$\frac{\partial^2 g}{\partial x^2}(x, y, z) = 6xyz$$

$$\frac{\partial^2 g}{\partial x \partial y}(x, y, z) = 3x^2 z$$

$$\frac{\partial^2 g}{\partial x \partial z}(x, y, z) = 3x^2 y$$

$$\frac{\partial^2 g}{\partial y \partial x}(x, y, z) = 3x^2 z$$

$$\frac{\partial^2 g}{\partial y^2}(x, y, z) = -4z^2$$

$$\frac{\partial^2 g}{\partial y \partial z}(x, y, z) = x^3 - 8yz$$

$$\frac{\partial^2 g}{\partial z \partial x}(x, y, z) = 3x^2 y$$

$$\frac{\partial^2 g}{\partial z \partial y}(x, y, z) = x^3 - 8yz$$

$$\frac{\partial^2 g}{\partial z^2}(x, y, z) = -4y^2$$

$$\begin{pmatrix} R_x \\ R_y \\ R_z \end{pmatrix} = d^2 g(x, y, z)(h, k) = d^2 g(x, y, z)(k, h)$$

$$2) \text{ Soit } H = (H_1, H_2) \in M_m \times M_m$$

$$dg(A+K_1, B+K_2)(H)$$

$$dg(AB)(H) = (AH_1 + H_1A)B + A^2 H_2$$

$$\text{Soit } K = (K_1, K_2)$$

$$\begin{aligned} dg(A+K_1, B+K_2)(H) &= ((A+K_1)H_1 + H_1(A+K_1))(B+K_2) + (A+K_1)^2 H_2 \\ &= (AH_1 + K_1 H_1 + H_1 A + H_1 K_1)(B+K_2) + (A^2 + AK_1 + K_1 A + K_1^2) H_2 \\ &= AH_1 B + AH_1 K_2 + K_1 H_1 B + K_1 H_1 K_2 + H_1 A B + H_1 A K_2 + H_1 K_1 B \\ &\quad + H_1 K_1 K_2 + A^2 H_2 + AK_1 H_2 + K_1 A H_2 + K_1^2 H_2 \end{aligned}$$

$$d^2 g(A, B)(H, K) = AH_1 K_2 + K_1 H_1 B + H_1 A K_2 + H_1 K_1 B + AK_1 H_2 + K_1 A H_2$$

$$\|dg(A+K_1, B+K_2)(H) - d^2 g(A, B)(H, K)\| = \|K_1 H K_2 + H_1 K_1 K_2 + K_1^2 H_2\|$$

$$\leq \|K_1 H_1 K_2\| + \|H_1 K_1 K_2\| + \|K_1^2 H_2\|$$

Exercice 8

Soit $H \in GL_m(\mathbb{R})$

$$1) g(A+H) = (A+H)^{-1}$$

$$x \mapsto \frac{1}{x}$$

$$dg(x)(h) = g'(x) \cdot h$$

$$= -\frac{h}{x^2}$$

$$= [A(I+A^{-1}H)]^{-1}$$

$$= (I+A^{-1}H)^{-1} A^{-1}$$

$$= [I - A^{-1}H + \sum_{R \geq 2} (-1)^R (A^{-1}H)^R] \cdot A^{-1}$$

$$= \underbrace{A^{-1}}_{g(A)} - \underbrace{A^{-1}HA^{-1}}_{\frac{1}{g(A)}} + \sum_{R \geq 2} \underbrace{(-1)^R (A^{-1}H)^R}_{\text{reste}} \cdot A^{-1}$$

$$\|g(A+H) - g(A) - \frac{1}{g(A)} H\|$$

$$= \left\| \sum_{R \geq 2} (-1)^R (A^{-1}H)^R A^{-1} \right\|$$

$$\leq \sum_{R \geq 2} \|A^{-1}\|^{R+1} \|H\|^R$$

$$\leq \|H\|^2 \cdot \|A^{-1}\|^3 \sum_{R=0}^{\infty} (\|A^{-1}\| \|H\|)^R$$

$$\leq \|H\|^2 \|A^{-1}\|^3 \frac{1}{1 - \|A^{-1}\| \|H\|}$$

$$\leq \|H\| \cdot \frac{\|A^{-1}\|^3 \|H\|}{1 - \|A^{-1}\| \|H\|} \rightarrow 0$$

2) Soit $A, B \in GL_n(\mathbb{R})$

$$\|d_g(A) - d_g(B)\| = \sup \|d_g(A)(H) - d_g(B)(H)\| \times \frac{1}{\|H\|}$$

$$\|d_g(A)(H) - d_g(B)(H)\| = \|B^{-1}HB^{-1} - A^{-1}HA^{-1}\|$$

$$= \|B^{-1}HB^{-1} - B^{-1}HA^{-1} + B^{-1}HA^{-1} - A^{-1}HA^{-1}\|$$

$$= \|B^{-1}H(B^{-1} - A^{-1}) + (B^{-1} - A^{-1})HA^{-1}\|$$

$$\leq \|B^{-1} - A^{-1}\| (\|B^{-1}H\| + \|HA^{-1}\|)$$

$$\leq \|B^{-1} - A^{-1}\| \cdot (\|B^{-1}\| + \|A^{-1}\|) \quad \text{car } \|B^{-1}HA^{-1}\| \rightarrow 0 \text{ quand}$$

$$\|A - B\| \rightarrow 0 \quad \text{puisque } g \text{ continue}$$

Exercice 6 - Suite

$$g(x, y) = g(x', y') \Rightarrow (x, y) = (x', y')$$

$$\begin{cases} x^3 + x = x'^3 + x' \\ y - x^2 = y' - x'^2 \end{cases} \quad \text{on } x \neq x' \quad \text{car } x^3 + x \nearrow$$

$$\Rightarrow y - x^2 = y' - x'^2 \Rightarrow y = y'$$

$$\text{Soit } (u, v) \in \mathbb{R}^2$$

$$\exists^! (x, y) \in \mathbb{R}^2 \text{ tel que } g(x, y) = (u, v)$$

$$g = x^3 + x$$

$$\begin{cases} x^3 + x = u \\ y - x^2 = v \end{cases}$$

$$g'(x) = 3x^2 + 1 > 0$$

$$g \nearrow \Rightarrow g \text{ surjective de } \mathbb{R}$$

dans $\mathbb{R}$

$$\Rightarrow y = v + x^2$$

$$= v + g^{-1}(u)^2$$

4) $dg(x, y)$ définie $Jg(x, y) = \begin{pmatrix} 3x^2+1 & 0 \\ -2x & 1 \end{pmatrix}$

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diff

$$\begin{vmatrix} 3x^2+1 & 0 \\ -2x & 1 \end{vmatrix} = 3x^2+1 > 0 \quad \forall x, y \in \mathbb{R}^2$$

donc $dg(x, y)$ est un isomorphisme donc d'après le théorème d'inversion globale g est C^1 difféomorphisme donc g^{-1} est C^1 .

Exercice 3

1) g n'est pas surjective car $(-1, 1, -1)$ n'a pas d'antécédent donc g n'est pas un C^1 difféomorphisme.

2) g injective de $] -\pi/2, \pi/2[$ dans $\mathbb{R}$

$$dg(x)(h) = \frac{1}{\cos^2(x)} h \text{ isomorphisme donc } g \text{ est } C^1 \text{ difféomorphisme}$$

3) • h est injective

• $h'(x) = 3x^2$ s'annule en 0 $dh(x)$ n'est pas un isomorphisme sur $\mathbb{R}$. Donc h n'est pas un C^1 difféomorphisme sur $\mathbb{R}$.

$\forall x \neq 0, h'(x) \neq 0$ donc h est un C^1 -difféomorphisme

local. On $h^{-1}(x) = x^{1/3}$

$$[h^{-1}(x)]' = \frac{1}{3} x^{-2/3} \text{ problème en } 0.$$

4) Soit (x, y) et (x', y')

$$R(x, y) = R(x', y') \Rightarrow \begin{cases} x = x' \\ y = y' \end{cases} \text{ donc } R \text{ injective}$$

$$J_R(x, y) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

det $J_R(x, y) = -2 \neq 0$ $dR(x, y)$ est un isomorphisme sur $\mathbb{R}^2$

Exercice 10 On cherche les points critiques tels que

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 0 \Leftrightarrow 3x^2 + 3y^2 - 15 = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \Leftrightarrow 6xy - 12 = 0 \end{cases}$$

$$x^2 + y^2 = 5 \rightarrow \text{équation de cercle}$$

$$xy = 2 \Leftrightarrow y = \frac{2}{x} \quad \text{On a } x^2 - 5x + 1 = 0$$

On trouve (1,2) (2,1) 4 points critiques

(-1,-2) (-2,-1)

$$A = \begin{pmatrix} \partial_{11}^2 f(x,y) = 6x & \partial_{12}^2 f(x,y) = 6y \\ \partial_{21}^2 f(x,y) = 6y & \partial_{22}^2 f(x,y) = 6x \end{pmatrix}$$

en (1,2): $\det(A) < 0$ niem

en (2,1): $\det(A) > 0 \Rightarrow p > 0 \Rightarrow \text{minimum}$

en (-1,-2): $\det(A) < 0$ niem

en (-2,-1): $\det(A) > 0 \Rightarrow p < 0 \Rightarrow \text{max}$

Exercice 11 $\partial_1 f(x,y) = 2x - y + 1$

$$\partial_2 f(x,y) = 2y - x + 1$$

Points critiques ?

$$\begin{cases} \partial_1 f(x,y) = 2x - y + 1 = 0 \\ \partial_2 f(x,y) = 2y - x + 1 = 0 \end{cases} \quad \begin{matrix} x = -1 \\ y = -1 \end{matrix}$$

$$\begin{vmatrix} \partial_{11}^2 f(x,y) = 2 & \partial_{12}^2 f(x,y) = -1 \\ \partial_{21}^2 f(x,y) = -1 & \partial_{22}^2 f(x,y) = 2 \end{vmatrix} = \begin{matrix} 3 > 0 \\ p > 0 \end{matrix} \quad \text{donc minimum local}$$

↓ bord à droite
 $g(y) = y^2 + y = g(y)$

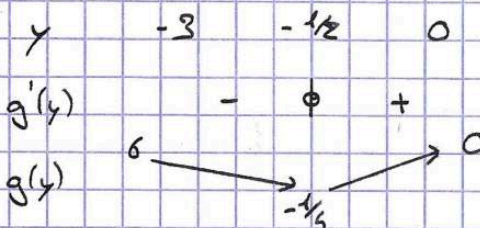
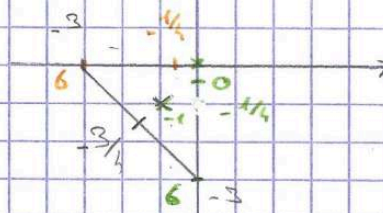
$$g'(y) = 2y + 1$$

$$x + y = 0$$

$$g(x, -x-3) = x^2 + (-x-3)^2 - x(-x-3) + x - x - 3$$

$$= x^2 + x^2 + 6x + 9 + x^2 + 3x + x - x - 3$$

$$= 3x^2 + 9x + 6 = h(x)$$



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d. 88

$$h'(x) = 6x + 5$$

$$3 \times \frac{5}{4} + 3 \left(-\frac{3}{2}\right) + 6$$

$$x \quad -3 \quad -\frac{3}{2} \quad 0$$

$$h'(x) \quad - \quad 0 \quad +$$



plus petite valeur = -1

plus grande = 6

Exercice 12

1) Libre de décision : me changent plus quand les deux sont égales : $20 + 5x = 40 \Rightarrow x = 4$

$$\begin{aligned} 2) D(x) &= [(20 + 5x) \times x \times 1000 + 40 \times (10 - x) \times 1000] \times \frac{1}{10000} \\ &= \frac{1}{10} (5x^2 + 20x + 400 - 40x) \\ &= \frac{1}{10} (5x^2 - 20x + 400) \end{aligned}$$

$$D'(x) = \frac{1}{10} (10x - 20)$$

$$D'(x) = 0 \Rightarrow x = 2$$

Exercice 13

Gm pose $g(x, y) = x^3 + y^3 - 2xy$

$$(1) \Rightarrow g(x, y) = 0$$

$$g: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$

$$\begin{aligned} g &\text{ est } C^1 \\ \mathcal{L}_1 &= \mathbb{R} \\ \mathcal{L}_2 &= \mathbb{R} \\ \mathcal{F} &= \mathbb{R} \\ \dim \mathcal{L}_2 &= \dim F \end{aligned}$$

$$a = (a_1, a_2) = (\underline{1}, \underline{1})$$

$$g(1, 1) = 0$$

$$\frac{\partial g}{\partial y}(x, y) = 3y^2 - 2x$$

$$d_2 g(x, y)(h_2) = (3y^2 - 2x) h_2$$

$$\frac{\partial g}{\partial y}(1, 1) = 1 \neq 0$$

donc $d_2 g(1, 1)$ est un isomorphisme de $\mathbb{R}$ dans $\mathbb{R}$

alors TFI

$\exists U_1$ voisinage de 1

$\exists U_2$ voisinage de 2

$\exists ! \varphi: U_1 \rightarrow U_2$ tel que

$$\forall x \in U_1, g(x, y) = 0 \Leftrightarrow y = \varphi(x)$$

$$g(x, \varphi(x)) = 0.$$

composée dans $\mathbb{R}$

$$\begin{aligned}\varphi'(x) &= (d_x g(x, \varphi(x)))^{-1} \circ d_x g(x, \varphi(x)) \\ &= \frac{3x^2 - 2\varphi(x)}{3\varphi^2(x) - 2x} \quad \varphi'(1) = 1\end{aligned}$$

3) $g(x, y) = x^3 + y^3 - 2xy$ On ne peut pas résoudre en x ou y en un point P si c'est un point critique

Point critique :

$$\begin{cases} 3x^2 - 2y = 0 \\ 3y^2 - 2x = 0 \end{cases} \quad \begin{aligned} y &= \frac{3}{2}x^2 \\ \frac{27}{4}x^4 - 2x &= 0 \\ x &= 0 \quad \text{ou} \quad x = \frac{2}{3} \\ y &= 0 \quad \quad y = \frac{2}{3} \end{aligned}$$

pas sur la courbe (1)

1) Équation de la tangente en (x_0, y_0)

$$(3x_0^2 - 2y_0)(x - x_0) + (3y_0^2 - 2x_0)(y - y_0) = 0$$

2) Tangente verticale $\Rightarrow x = \text{cte}$ donc on doit avoir $d_x g(x, y) = 0$ ie $3y_0^2 - 2x_0 = 0$

$\rightarrow$ nouvelle équation tangente $\frac{(3x_0^2 - 2y_0)(x - x_0)}{\neq 0} = 0$

$$\Rightarrow x = x_0 = \text{cte}.$$

Exercice 14

1) Soit $H \in \mathcal{M}_m(\mathbb{R})$

$$\begin{aligned}g(X, Y+H) &= B - \frac{(I_m - X + (Y+H))^2}{2} \\ &= B - \frac{(I_m - X + Y^2 + YH + HY + H^2)}{2} \\ &= B - \frac{I_m - X + Y^2}{2} - \frac{YH + HY}{2} - \frac{H^2}{2}\end{aligned}$$

2) $\lim I_m$, on doit avoir $\varphi(I_m)^2 = I_m$

on se place en (I_m, I_m)

$$g(X, Y) = g(X, Y) - (B - I_m/2) = \frac{X - Y^2}{2}$$

$$g(I_m, I_m) = 0 \quad d_x g(X, Y)(H) = \frac{YH + HY}{2}$$

$d_2 g(I_m, I_m) = -H$ donc $d_2 g(I_m, I_m)$ est un isomorphisme de $M_m(\mathbb{R})$ dans $M_m(\mathbb{R})$

$\exists U_1$ tel que $I_m \in U_1$

$\exists ! \varphi : U_1 \rightarrow U_2$ tel que

$\exists U_2$ tel que $I_m \in U_2$

$$g(X, Y) = 0 \Leftrightarrow Y = \varphi(X)$$

$$\Leftrightarrow \frac{X - \varphi(X)^2}{2} = 0$$

3) φ n'est pas unique on peut

prendre $-\varphi$ pour résoudre le problème

Exercice 15

$$1) z = c = (1 - 8a(a^2 + b^2))(x - a) - 8b(a^2 + b^2)(y - b)$$

$$\text{Plan tangent horizontal} \begin{cases} 1 - 8a(a^2 + b^2) = 0 & 1 - 8a^3 = 0 & a = 1/2 \\ 8b(a^2 + b^2) = 0 & b = 0 \end{cases}$$

$$(a, b, c) \in S$$

$$c = a - 2(a^2 + b^2)^2$$

$$= 3/8$$

$$2) g(x, y) = f(x, y) - [c + (1 - 8a(a^2 + b^2))(x - a) + (-8b(a^2 + b^2))(y - b)]$$

Étude du signe en $(0, 0, c)$

$$g(x, y) = f(x, y) - [x]$$

$$= -2(x^2 + y^2) \leq 0 \quad \text{donc } S \text{ est au-dessus du plan tangent}$$

3) Par définition

$$\frac{dg}{dx}(x, y) = \frac{\partial g}{\partial x}(x, y) - \frac{\partial g}{\partial x}(a, b)$$

$$\frac{dg}{dy}(x, y) = \frac{\partial g}{\partial y}(x, y) - \frac{\partial g}{\partial y}(a, b)$$

en (a, b)

$$\frac{\partial g}{\partial x}(a, b) = 0$$

Par construction (a, b) est un pt critique de g .

$$\frac{\partial g}{\partial y}(a, b) = 0$$

$$\frac{\partial^2 g(x)}{\partial x^2}(x,y) = \frac{\partial^2 g}{\partial x^2}(x,y) = -8(3x^2 + y^2)$$

$$\frac{\partial^2 g(x)}{\partial y \partial x} = -16xy$$

$$\frac{\partial^2 g(x,y)}{\partial y^2} = -8(x^2 + 3y^2)$$

$$\begin{pmatrix} -8(3x^2 + y^2) & -16xy \\ -16xy & -8(x^2 + 3y^2) \end{pmatrix} = H(x,y)$$

$$\det H(x,y) = 192(x^4 + 8x^2y^2 + y^4)$$

$$= 192(x^2 + y^2)^2 \geq 0$$

$$\text{pour } (x,y) \neq (0,0)$$

donc g est un max local donc S est en dessous de plan tangent

Exercice 1

$$\begin{aligned}
 1) c_m(g) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) e^{-imx} dx \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ax} \cdot e^{-imx} dx \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(a-im)x} dx
 \end{aligned}$$

$$\begin{aligned}
 &\bullet \text{ si } a \notin i\mathbb{Z} \quad (a \neq im, m \in \mathbb{Z}) \\
 c_m(g) &= \frac{1}{2\pi} \left[\frac{1}{a-im} e^{(a-im)x} \right]_{-\pi}^{\pi} = \frac{1}{2\pi} \left(\frac{1}{a-im} (e^{(a-im)\pi} - e^{-(a-im)\pi}) \right) \\
 &= \frac{(1)^m}{2\pi(a-im)} (e^{a\pi} - e^{-a\pi}) \\
 &= \frac{(-1)^m \operatorname{sh}(a\pi)}{\pi(a-im)}
 \end{aligned}$$

$$\bullet \text{ si } a \in i\mathbb{Z} \\
 c_m(g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-m)x} dx = \begin{cases} 1 & \text{si } m = m \\ 0 & \text{si } m \neq m \end{cases}$$

Exercice 2

g 2π périodique

$\forall x \in]-\pi, \pi[$, $g(x) = \operatorname{ch}(ax)$

1) $a \in i\mathbb{Z}$ $\Leftrightarrow \exists m \in \mathbb{Z}$ tel que $a = im$

$$\begin{aligned}
 c_m &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{ax} + e^{-ax}}{2} e^{-imx} dx \\
 &= \frac{1}{4\pi} \int_{-\pi}^{\pi} e^{(a-im)x} + e^{-(a+im)x} dx \\
 &= \frac{1}{4\pi} \int_{-\pi}^{\pi} (e^{i(m-m)x} + e^{-i(m+m)x}) dx
 \end{aligned}$$

$$\text{si } m = m \quad c_m(g) = 1/2$$

$$m \neq -m \quad c_m(g) = 1/2$$

$$m \neq m \quad c_m(g) = 0$$

2) $a \in \mathbb{R}_+^*$

$$c_m(g) = \frac{1}{4\pi} \int_{-\pi}^{\pi} e^{(a-im)x} + e^{-(a+im)x} dx$$

$$\begin{aligned}
&= \frac{1}{4\pi} \left[\frac{e^{(a-im)x}}{(a-im)} - \frac{e^{-(a+im)x}}{(a+im)} \right]_{-\pi}^{\pi} \\
&= \frac{1}{4\pi} \left(\frac{e^{(a-im)\pi} - e^{-(a-im)\pi}}{a-im} - \frac{e^{-(a+im)\pi} - e^{(a+im)\pi}}{a+im} \right) \\
&= \frac{(-1)^m}{2\pi} \left(\frac{\sinh(a\pi)}{a-im} + \frac{\sinh(a\pi)}{a+im} \right) = \frac{(-1)^m \sinh(a\pi) a}{\pi(a^2+m^2)}
\end{aligned}$$

Exercice 3 8) La fonction n'est pas périodique continue donc

$$g \notin \mathcal{C}_{2\pi}$$

On ne peut pas appliquer Parseval.

$$g) c_m(g) = \frac{(-1)^m \sinh(a\pi)}{\pi(a^2+m^2)}$$

Parseval

$$\begin{aligned}
\|g\|_2^2 &= \sum_{m=-\infty}^{+\infty} \left| \frac{(-1)^m a \sinh(a\pi)}{\pi(a^2+m^2)} \right|^2 \\
&= \frac{a^2 \sinh^2(a\pi)}{\pi^2} \sum_{m=-\infty}^{+\infty} \frac{1}{(a^2+m^2)^2} \\
\|g\|_2^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\operatorname{ch}(ax)|^2 dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{2ax} + 2 + e^{-2ax}}{2} dx \\
&= \frac{1}{2\pi} \left[\frac{1}{2a} e^{2ax} + 2x - \frac{1}{2a} e^{-2ax} \right]_{-\pi}^{\pi} \\
&= \frac{1}{2\pi} \left(\frac{2\sinh(2a\pi)}{a} + 4\pi \right) \\
&= \frac{\sinh(2a\pi)}{a\pi} + \frac{1}{2} \\
\Rightarrow \frac{\sinh(2a\pi)}{a\pi} + \frac{1}{2} &= \left(\frac{a \sinh(a\pi)}{\pi} \right)^2 \sum_{m=-\infty}^{+\infty} \frac{1}{(a^2+m^2)^2} \\
\Rightarrow \sum_{m=-\infty}^{+\infty} \frac{1}{a^2+m^2} &= \left(\frac{\pi}{a \sinh(a\pi)} \right)^2 \left(\frac{\sinh(2a\pi)}{a\pi} + \frac{1}{2} \right)
\end{aligned}$$

S. a = im (a ∈ iZ)

$$\begin{aligned}
\text{si } m=m & c_m(g) = 1 \\
m \neq m & c_m(g) = 0
\end{aligned}$$

$$\text{si } p < |m| \quad s_p(g) = 0$$

$$p \geq |m| \quad s_p(g) = c_m$$

S. a ≠ im

$$c_m(g) = \frac{(-1)^m \sinh(a\pi)}{\pi(a-im)}$$

$$s_p(g) = \sum_{m=-p}^p \frac{(-1)^m \sinh(a\pi)}{\pi(a-im)} e_m$$

$$2) \text{ si } p < |m| \quad z_p(g) = 0$$

$$p \geq |m| \quad z_p(g) = \frac{1}{2} e_{-m} + \frac{1}{2} e_m = \gamma_m$$

3
d'g

$$3) \text{ a } \in \mathbb{R}^{+*}$$

$$z_p(g) = \sum_{m=-p}^p \frac{a(-1)^m \operatorname{sh}(am)}{\pi(a^2 + m^2)} e_m$$

1) Est ce que $g \in \mathcal{D}_{2\pi}$?

$$\left. \begin{array}{l} \lim_{x \rightarrow \pi^-} g(x) = e^{a\pi} \\ \lim_{x \rightarrow \pi^+} g(x) = e^{-a\pi} \end{array} \right\} \frac{g(x^+) + g(x^-)}{2} = \operatorname{ch}(ax) \quad \text{dome } g \in \mathcal{D}_{2\pi}$$

Exercice 3 suite on a avec dans le cours

$$3) 1) \quad g = g_{\pi}$$

$$\begin{aligned} \|g\|_2^2 &= \|g\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(t)|^2 dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |e^{at}|^2 dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{2ut} dt \end{aligned}$$

$$\text{où } u = \operatorname{Re}(a)$$

$$\text{Si } u = \operatorname{Re}(a) = 0$$

$$\|g\|_2^2 = 1$$

$$\text{Si } u \neq 0$$

$$\begin{aligned} \|g\|_2^2 &= \frac{1}{2\pi} \left[\frac{1}{2u} e^{2ut} \right]_{-\pi}^{\pi} \\ &= \frac{\operatorname{sh}(2u\pi)}{2u\pi} \end{aligned}$$

Formule de Parseval

$$\begin{aligned} \sum_{m=-\infty}^{+\infty} |c_m(g)|^2 &= \sum_{m=-\infty}^{+\infty} \left| \frac{(-1)^m \operatorname{sh}(am)}{\pi(a - im)} \right|^2 \\ &= \sum_{m=-\infty}^{+\infty} \left| \frac{1}{a - im} \right|^2 \times \frac{|\operatorname{sh}(am)|^2}{\pi^2} \end{aligned}$$

$$\text{Si } \operatorname{Re}(a) = 0$$

$$\frac{\sin(u\pi)^2}{\pi^2} \sum_{m=-\infty}^{+\infty} \frac{1}{|v - m|^2} = 1 \quad \text{et donc} \quad \sum_{m=-\infty}^{+\infty} \frac{1}{|v - m|^2} = \frac{\pi^2}{\sin(u\pi)^2}$$

Si $\operatorname{Re}(a) \neq 0$

$$\frac{|\operatorname{sh}(a\pi)|^2}{\pi^2} \sum_{n=-\infty}^{+\infty} \frac{1}{|a - in|^2} = \frac{\operatorname{sh}(2\operatorname{Re}(a)\pi)}{2\operatorname{Re}(a)\pi}$$

$$\sum_{n=-\infty}^{+\infty} \frac{1}{|a - in|^2} = \frac{\operatorname{sh}(2\operatorname{Re}(a)\pi)\pi}{2\operatorname{Re}(a)|\operatorname{sh}(a\pi)|^2}$$

Exercice 4

1) On ne peut pas appliquer le théorème de convergence normale ($g \notin \mathcal{E}_{\text{nm}}$)

On peut appliquer le théorème de convergence simple donc

$\sum c_m(g) e_m$ converge vers g_n .

$\forall x \in \mathbb{R} \quad a \notin i\mathbb{Z}$

$$g_n(x) = \sum_{m=-\infty}^{+\infty} \frac{(-1)^m \operatorname{sh}(a\pi)}{\pi(a - im)} e^{imx}$$

$\forall x \in]-\pi, \pi[$

$$e^{ax} = \sum_{m=-\infty}^{+\infty} \frac{(-1)^m \operatorname{sh}(a\pi)}{\pi(a - im)} e^{imx}$$

Pour $x=0$

$$\Rightarrow \sum_{m=-\infty}^{+\infty} \frac{(-1)^m}{a - im} = \frac{\pi}{\operatorname{sh}(a\pi)}$$

Pour $x=\pi$

$$\Rightarrow \sum_{m=-\infty}^{+\infty} \frac{1}{a - im} = \pi \coth(a\pi)$$

2) $g \in \mathcal{E}_{\text{nm}}$ On peut appliquer le théorème de convergence normale sur $\mathbb{R}$.

$\forall x \in \mathbb{R}$

$$g(x) = \sum_{m=-\infty}^{+\infty} \frac{a(-1)^m \operatorname{sh}(a\pi)}{\pi(a^2 + m^2)} e^{imx}$$

Pour $x=0$

$$g(x) = 1 = \sum_{m=-\infty}^{+\infty} \frac{a(-1)^m \operatorname{sh}(a\pi)}{\pi(a^2 + m^2)} e^{imx}$$

$$\sum_{m=-\infty}^{+\infty} \frac{(-1)^m}{a^2 + m^2} = \frac{\pi}{a \operatorname{sh}(a\pi)}$$

Pour $x = \pi$

$$\operatorname{ch}(a\pi) = \sum_{n=-\infty}^{+\infty} \frac{e^{2n\pi a}}{\pi(a^2 + n^2)}$$

$$\sum \frac{1}{a^2 + n^2} = \frac{\pi}{a} \coth(a\pi)$$

Exo 6

Trouve la Période $\Rightarrow f \in E_T$

$\Rightarrow$ moyenne quadratique $\Rightarrow$ Parseval $\rightarrow \|g\|_2^2 = \sum |c_m|^2$

Exercice 5

$$f(x) = \left| \sin\left(\frac{\pi}{T} x\right) \right|$$

$$\sin\left(\frac{\pi}{T}(x+\alpha)\right) = \sin\left(\frac{\pi}{T}x + \frac{\alpha\pi}{T}\right)$$

$$\frac{\alpha\pi}{T} = 2\pi \Leftrightarrow \alpha = 2T$$

T est une période (valeur absolue)

$$f(x+T) = \left| \sin\left(\frac{\pi}{T}(x+T)\right) \right| = f(x)$$

Soit $\alpha \in \mathbb{R}$ tel que $\begin{cases} 0 < \alpha < T \\ f(x+\alpha) = f(x) \end{cases}$
 $\forall x \in \mathbb{R}$

Pour $x = 0$

$$\left. \begin{aligned} f(\alpha) = f(0) = 0 & \quad \left| \sin\left(\frac{\pi}{T}\alpha\right) \right| = 0 \\ \text{et } 0 < \frac{\pi}{T}\alpha < \pi \end{aligned} \right\} \text{ Impossible}$$

Donc $f \in E_T \Rightarrow$ moyenne quadratique $\lim_{p \rightarrow +\infty} \|f - S_p(f)\|_2 = 0$

$\rightarrow$ formule de Parseval $\|f\|_2^2 = \sum_{n=-\infty}^{+\infty} |c_n(f)|^2$

Calcul de $c_n(f)$

$$\begin{aligned} c_n(f) &= \frac{1}{T} \int_0^T \sin\left(\frac{\pi}{T}x\right) \exp\left(-\frac{2i\pi n}{T}x\right) dx \\ &= \frac{1}{T} \int_0^T \frac{e^{\frac{i\pi x}{T}} - e^{-\frac{i\pi x}{T}}}{2i} \exp\left(-\frac{2i\pi n}{T}x\right) dx \\ &= \frac{1}{2iT} \int_0^T \exp\left(\frac{i\pi}{T}(1-2n)x\right) - \exp\left(-\frac{i\pi}{T}(1+2n)x\right) dx \\ &= \frac{1}{2iT} \left[\frac{T e^{\frac{i\pi}{T}(1-2n)x}}{i\pi(1-2n)} + \frac{T e^{-\frac{i\pi}{T}(1+2n)x}}{i\pi(1+2n)} \right]_0^T \\ &= \frac{1}{2i} \left(\frac{e^{i\pi(1-2n)} - 1}{i\pi(1-2n)} + \frac{e^{-i\pi(1+2n)} - 1}{i\pi(1+2n)} \right) \\ &= \frac{2}{\pi(1-n^2)} \end{aligned}$$

$$\begin{aligned} \text{Calcul de } \|f\|_2^2 & \quad \|f\|_2^2 = \frac{1}{T} \int_0^T \sin^2\left(\frac{\pi}{T}x\right) dx \\ &= \frac{1}{T} \int_0^T \frac{1 - \cos\left(\frac{2\pi}{T}x\right)}{2} dx \\ &= 1/2 \end{aligned}$$

Au final Poserel:

$$\sum_{m=-\infty}^{+\infty} \frac{2}{\pi(1-4m^2)} = \frac{1}{2}$$

$$\sum_{m=-\infty}^{+\infty} \frac{1}{1-4m^2} = \frac{\pi^2}{8}$$

On a $g \in \mathcal{E}_T$ et $\mathcal{E}^1$ par Morceaux

Donc $(S_p(g))$ converge normalement vers g sur $\mathbb{R}$

En particulier: $\forall x \in \mathbb{R}$, $g(x) = \sum_{m=-\infty}^{+\infty} \frac{2}{\pi(1-4m^2)} \exp\left(\frac{2im}{T} mx\right)$

Pour $x=0$

$$\sum_{m=-\infty}^{+\infty} \frac{2}{\pi(1-4m^2)} = 0$$

$$\sum_{m=-\infty}^{+\infty} \frac{1}{1-4m^2} = 0$$

Pour $x=T/2$

$$1 = \sum_{m=-\infty}^{+\infty} \frac{2(-1)^m}{\pi(1-4m^2)}$$

$$\sum_{m=-\infty}^{+\infty} \frac{(-1)^m}{1-4m^2} = \frac{\pi}{2}$$

Exercice 6

$$1) c_m(g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\pi-x}{2} e^{-imx} dx$$

$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} (\pi-x) e^{-imx} dx$$

$$= \frac{1}{4\pi} \left(\frac{\pi-x}{im} e^{-imx} \Big|_{-\pi}^{\pi} - \frac{1}{im} \int_{-\pi}^{\pi} e^{-imx} dx \right)$$

$$\underline{m \neq 0} = \frac{(-1)^m}{2im}$$

$$\underline{m=0} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\pi-x}{2} dx$$

$$= \frac{1}{4\pi} \left[\pi x - \frac{x^2}{2} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{4\pi} (\pi^2 + \pi^2)$$

$$= \frac{1}{2\pi} \cdot \frac{\pi^2}{2}$$

2)

$$c_m = c_m + c_{-m}$$

$$= 0$$

$$b_m = i(c_m - c_{-m})$$

$$= \cancel{0} \cdot \frac{1}{im}$$

$$3) f(\pi) = 0$$

$$\lim_{p \rightarrow +\infty} S_p(f)(\pi) \neq f(\pi) \\ = \pi/2$$

$$S_p(f) = \sum_{m=-p}^{+p} c_m(f) e_m$$

$$4) f \in \mathcal{C}^1_{2\pi}$$

f n'est pas continue $f \notin \mathcal{C}_{2\pi}$

$f \neq \sum g_n$ donc $f \notin \mathcal{D}_{2\pi}$

f est $\mathcal{E}^1$ par morceaux

$$\text{Dirichlet: } \forall x \in \mathbb{R}, f_n(x) = \sum_{m=-\infty}^{+\infty} c_m(f) e_m(x) \\ = c_0(f) + \sum_{m=1}^{+\infty} c_m(f) e_m(x) \quad \text{car } c_m(f) = 0 \\ = \frac{c_0(f)}{2} + \sum_{m=1}^{+\infty} b_m(f) \sin(mx) \\ \forall x \in \mathbb{R} \quad f_n(x) = \frac{\pi}{2} + \sum_{m=1}^{+\infty} \frac{(-1)^m}{m} \sin(mx)$$

On prend $x = \pi/2$

$$f_n(\pi/2) = f(\pi/2) = \frac{\pi - \pi/2}{2} = \pi/4$$

$$\frac{\pi}{4} = \frac{\pi}{2} - \sum_{p=0}^{+\infty} \frac{(-1)^p}{2p+1} \Rightarrow \sum_{p=0}^{+\infty} \frac{(-1)^p}{2p+1} = \frac{\pi}{4}$$

$$5) \|f\|_2^2 = \left| \frac{c_0(f)}{2} \right|^2 + \frac{1}{2} \sum_{m=1}^{+\infty} |a_m(f)|^2 + |b_m(f)|^2 \quad (\text{Parseval})$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{\pi-x}{2} \right)^2 dx = \frac{1}{8\pi} \int_{-\pi}^{\pi} (\pi-x)^2 dx \\ = \frac{\pi^3}{6}$$

$$\frac{\pi^3}{6} = \frac{\pi^3}{4} + \frac{1}{2} \sum_{m=1}^{+\infty} \frac{1}{m^2}$$

$$\frac{\pi^3}{6} = \sum_{m=1}^{+\infty} \frac{1}{m^2}$$

Exercice 7

$$f(x+1) = x+1 - \zeta^-(x) \\ = x+1 - \zeta^-(x) - 1 \\ = f(x)$$

Supposons qu'il existe $\alpha \in \mathbb{R}^{++}$ tel que

$$0 < \alpha < 1$$

$$f(x+\alpha) = f(x) \quad \forall x \in \mathbb{R}$$

Pour $x=0$

$$\alpha = f(\alpha) = f(0) = 0$$

impossible donc $T=1$

$$f \in E_{\mathcal{H}_1}$$

$$f \notin E_1$$

$$f \in \mathcal{D}_1$$

$f \in \mathcal{E}_1$ par morceaux

$$\begin{aligned} c_m(f) &= \frac{1}{1} \int_0^1 f(x) \exp\left(-\frac{2i\pi m}{1} x\right) dx \\ &= \int_0^1 x e^{-2i\pi m x} dx \end{aligned}$$

$$m=0 \quad c_0(f) = \int_0^1 x dx = 1/2$$

$$m \neq 0 \quad c_m(f) = \left[-\frac{x}{2i\pi m} e^{-2i\pi m x} \right]_0^1 + \frac{1}{2i\pi m} \int_0^1 e^{-2i\pi m x} dx$$

$$a_0(f) = 1$$

$$b_m(f) = -\frac{1}{\pi m}$$

$$a_m(f) = 0$$

Théorème de convergence quadratique $\Rightarrow$ Parseval

Théorème de convergence simple $\Rightarrow$ Dirichlet (ou voir f_n)