Outils Mathématiques – Champ scalaires et vectoriels

CSV1

1)

$$\overrightarrow{grad}(GK) = \frac{\partial(GK)}{\partial x}\vec{e}_x+\frac{\partial(GK)}{\partial y}\vec{e}_y+\frac{\partial(GK)}{\partial z}\vec{e}_z$$
$$= (K\frac{\partial G}{\partial x}+G\frac{\partial K}{\partial x})\vec{e}_x+(K\frac{\partial G}{\partial y}+G\frac{\partial K}{\partial y})\vec{e}_y+(K\frac{\partial G}{\partial z}+G\frac{\partial K}{\partial z})\vec{e}_z$$
$$= K(\frac{\partial G}{\partial x}\vec{e}_x+\frac{\partial G}{\partial y}\vec{e}_y+\frac{\partial G}{\partial z}\vec{e}_z)+G(\frac{\partial K}{\partial x}\vec{e}_x+\frac{\partial K}{\partial y}\vec{e}_y+\frac{\partial K}{\partial z}\vec{e}_z)$$
$$\overrightarrow{grad}(GK) = K\,\overrightarrow{grad}(G)+G\,\overrightarrow{grad}(K)$$

2) Coordonnées sphérq → $\overrightarrow{grad}(\Phi(r,\theta,\varphi)) = \frac{\partial\Phi}{\partial r}\vec{e}_r+\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\vec{e}_\theta+\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\varphi}\vec{e}_\varphi$

- $\overrightarrow{grad}(r) = \vec{e}_r = \frac{\vec{r}}{r}$
- $\overrightarrow{grad}(\frac{1}{r}) = -\frac{1}{r^2}\vec{e}_r = -\frac{\vec{r}}{r^3}$
- $\overrightarrow{grad}(\ln r) = \frac{1}{r}\vec{e}_r = \frac{\vec{r}}{r^2}$
- $\overrightarrow{grad}(r\ln r) = (\ln r+r\frac{1}{r})\vec{e}_r = (\frac{\ln(r)+1}{r})\vec{r}$

<u>Rmq</u> : $\overrightarrow{grad}(r\ln r) = \ln(r)\overrightarrow{grad}(r)+r\,\overrightarrow{grad}(\ln r) = \ln(r)\frac{\vec{r}}{r}+r\frac{\vec{r}}{r^2} = (\frac{\ln r}{r}+\frac{1}{r})\vec{r} = (\frac{\ln(r)+1}{r})\vec{r}$

CSV 2

$$\overrightarrow{grad}f = \frac{\partial}{\partial x}(\frac{z^2}{\sqrt{x^2+y^2+z^2}})\vec{e}_x+\frac{\partial}{\partial y}(\frac{z^2}{\sqrt{x^2+y^2+z^2}})\vec{e}_y+\frac{\partial}{\partial z}(\frac{z^2}{\sqrt{x^2+y^2+z^2}})\vec{e}_z$$

- $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[z^2(x^2+y^2+z^2)^{-1/2}] = z^2(-\frac{1}{2})(2x)(x^2+y^2+z^2)^{-3/2} = -xz^2(x^2+y^2+z^2)^{-3/2}$
- $\frac{\partial f}{\partial y} = -yz^2(x^2+y^2+z^2)^{-3/2}$
- $\frac{\partial f}{\partial z} = 2z(x^2+y^2+z^2)^{-1/2}-zz^2(x^2+y^2+z^2)^{-3/2}$

$$\overrightarrow{grad}f = -xz^2(x^2+y^2+z^2)^{-3/2}\vec{e}_x-yz^2(x^2+y^2+z^2)^{-3/2}\vec{e}_y+[2z(x^2+y^2+z^2)^{-1/2}-zz^2(x^2+y^2+z^2)^{-3/2}]\vec{e}_z$$
$$\overrightarrow{grad}f = 2z(x^2+y^2+z^2)^{-1/2}\vec{e}_z\ -z^2(x^2+y^2+z^2)^{-3/2}(x\vec{e}_x+y\vec{e}_y+z\vec{e}_z)$$

Calcul avec $\overrightarrow{grad}f = \overrightarrow{grad}(GK)$, $f(M) = \frac{z^2}{r}$, $r = \sqrt{x^2+y^2+z^2}$

$$\overrightarrow{grad}(\frac{z^2}{r}) = \frac{1}{r}\overrightarrow{grad}(z^2)+z^2\overrightarrow{grad}(\frac{1}{r}) = \frac{1}{r}2z\vec{e}_z+z^2(\frac{-1}{r^2})\vec{e}_r = \frac{2z}{r}\vec{e}_z-z^2\frac{\vec{r}}{r^3}$$
$$= \frac{2z}{\sqrt{x^2+y^2+z^2}}\vec{e}_z-z^2\frac{(x\vec{e}_x+y\vec{e}_y+z\vec{e}_z)}{(\sqrt{x^2+y^2+z^2})^3}$$
$$\overrightarrow{grad}(\frac{z^2}{r}) = 2z(x^2+y^2+z^2)^{-1/2}\vec{e}_z\ -z^2(x^2+y^2+z^2)^{-3/2}(x\vec{e}_x+y\vec{e}_y+z\vec{e}_z)$$

CSV4

$C(\vec{B}(M)) = \oint_\Gamma \vec{B}.d\overrightarrow{OM}$

- Calcul de $\vec{B}.d\overrightarrow{OM} = B_x dx + B_y dy + B_z dz$

  Or ici, $d\overrightarrow{OM} = dx\,\vec{e}_x + dy\,\vec{e}_y$, donc $\vec{B}.d\overrightarrow{OM} = B_x dx + B_y dy = (2x - y)dx + (x+y)dy$

  Donc $C(\vec{B}) = \oint_\Gamma (2x - y)dx + (x+y)dy$

- Paramétrage de Γ

$$M(\Gamma)\begin{cases} x(p) = R\cos p = 3\cos p \\ y(p) = R\sin p = 3\sin p \quad p\in[0,2\pi] \\ z(p) = 0 \end{cases}$$

$dx = -3\sin(p)dp$

$dy = 3\cos(p)dp$

- On injecte dans l'express°

$$\begin{aligned} \Rightarrow (2x - y)dx + (x+y)dy &= (6\cos p - 3\sin p)(-3\sin(p)dp) + (3\cos p + 3\sin p)(3\cos(p)dp) \\ &= [-18\cos p\sin p + 9\sin^2 p + 9\cos^2 p + 9\cos p\sin p]dp \\ &= (-9\cos p\sin p + 9)dp \\ &= 9(1 - \cos p\sin p)dp \end{aligned}$$

d'où $C(\vec{B}) = \int_0^{2\pi} 9(1 - \cos p\sin p)dp = 9[p - \frac{\sin^2 p}{2}]_0^{2\pi} = 18\pi$

CSV5

- $\vec{A}.d\overrightarrow{OM} = 3xy\,dx - 5z\,dy + 10x\,dz$
- M(Γ) $\begin{cases} x(t) = 1+t^2 \\ y(t) = 2t^2 \quad t\in[0,1] \\ z(t) = t^3 \end{cases}$ $\quad dx = 2t\,dt$, $dy = 4t\,dt$, $dz = 3t^2 dt$
- $3xy\,dx - 5z\,dy + 10x\,dz = (12t^5 + 10t^4 + 12t^3 + 30t^2)dt$

$C(\vec{A}) = \int_\Gamma \vec{A}.d\overrightarrow{OM} = \int_0^1 (12t^5 + 10t^4 + 12t^3 + 30t^2)dt = [2t^6 + 2t^5 + 3t^4 + 10t^3]_0^1 = 17$

CSV6

1) $\vec{F}(M) = xy\,\vec{e}_x - z^2\vec{e}_y - x^2\vec{e}_z$

- $W_{OABP}(\vec{F}) = \int_{\widehat{OABP}} \vec{F}.d\overrightarrow{OM} = \int_{\widehat{OA}} \vec{F}.d\overrightarrow{OM} + \int_{\widehat{AB}} \vec{F}.d\overrightarrow{OM} + \int_{\widehat{BP}} \vec{F}.d\overrightarrow{OM}$

  - $\int_{\widehat{OA}} \vec{F}.d\overrightarrow{OM} \rightarrow d\overrightarrow{OM} = dx\,\vec{e}_x \Rightarrow \vec{F}.d\overrightarrow{OM} = xy\,dx$

    $M\in[OA]\begin{cases} x\in[0,x_0] \\ y = 0 \\ z = 0 \end{cases} \Rightarrow \int_{\widehat{OA}} \vec{F}.d\overrightarrow{OM} = 0$

  - $\int_{\widehat{AB}} \vec{F}.d\overrightarrow{OM} \rightarrow d\overrightarrow{OM} = dy\,\vec{e}_y \Rightarrow \vec{F}.d\overrightarrow{OM} = -z^2 dy$

    $M\in[AB]\begin{cases} x_0 \\ y\in[0,y_0] \\ z = 0 \end{cases} \Rightarrow \int_{\widehat{AB}} \vec{F}.d\overrightarrow{OM} = 0$

- $\int_{\widehat{BP}} \vec{F}.d\overrightarrow{OM} \quad \rightarrow \quad d\overrightarrow{OM} = dz\,\vec{e}_z \quad \Rightarrow \quad \vec{F}.d\overrightarrow{OM} = -x^2 dz$

$$M \in [BP] \begin{cases} x_0 \\ y_0 \\ z \in [0, z_0] \end{cases} \quad \Rightarrow \quad \int_{\widehat{BP}} \vec{F}.d\overrightarrow{OM} = \int_0^{z_0} -x_0^2 dz = -x_0^2 z_0$$

$\Rightarrow \quad W_{OABP}(\vec{F}) = -x_0^2 z_0$

- $W_{OCDP}(\vec{F}) = \int_{\widehat{OC}} \vec{F}.d\overrightarrow{OM} + \int_{\widehat{CD}} \vec{F}.d\overrightarrow{OM} + \int_{\widehat{DP}} \vec{F}.d\overrightarrow{OM} = 0+0+\frac{x_0^2 y_O}{2} = \frac{x_0^2 y_O}{2}$
- $W_{OCBP}(\vec{F}) = \int_{\widehat{OC}} \vec{F}.d\overrightarrow{OM} + \int_{\widehat{CB}} \vec{F}.d\overrightarrow{OM} + \int_{\widehat{BP}} \vec{F}.d\overrightarrow{OM} = 0+\frac{x_0^2 y_O}{2} - x_0^2 z_0 = \frac{x_0^2 y_O}{2} - x_0^2 z_0$

2) $W_{OABP} \neq W_{OCDP} \neq W_{OCBP} \quad \Rightarrow \quad \nexists f(M) / \vec{F} = \overrightarrow{grad} f$

$\Leftrightarrow \overrightarrow{rot}(\vec{F})$ $n'est\ pas = \vec{0} \quad \forall (x, y, z)$

$\overrightarrow{rot}(\vec{F}) = 2z\,\vec{e}_x + 2x\,\vec{e}_y - x\,\vec{e}_z \quad \neq \vec{0} \quad \forall (x, y, z)$

Notes :
-q : -ique(s)
rmq : remarque(s)
° : -ion(s)