Outils Mathématiques – Calcul Intégral

Intégrales Curvilignes

IC1

$$I(f) = \int_{\widehat{AB}} f(M)\,ds \quad\rightarrow\quad f(M) = \frac{1}{x-y}\quad \text{, et } \Gamma : \quad y = \frac{x}{2} - 2 \quad \text{sur} \begin{cases} A(0,-2) \\ B(4,0) \end{cases}$$

- $$ds = \sqrt{dx^2+dy^2} = \sqrt{1+(\frac{dy}{dx})^2}\,dx$$

  $$\frac{dy}{dx} = \frac{1}{2} \Rightarrow ds = \sqrt{1+(\frac{1}{2})^2}\,dx = \frac{\sqrt{5}}{2}\,dx$$

- $$f(M) = \frac{1}{x-y} = \frac{1}{x-(x/2-2)} = \frac{1}{x/2+2} = \frac{2}{x+4}$$

- $$I(f) = \int_{x_A=0}^{x_B=4} \frac{2}{x+4}\frac{\sqrt{5}}{2}dx = \sqrt{5}\int_0^4 \frac{1}{x+4}dx = \sqrt{5}\,[\ln(x+4)]_0^4 = \sqrt{5}\,(\ln 8 - \ln 4) = \sqrt{5}\,\ln 2$$

IC2

1a) $$ds = \sqrt{1+(\frac{dy}{dx})^2}\,dx \quad\rightarrow\quad \frac{dy}{dx} = \frac{2x}{4} - \frac{1}{2x} = \frac{x^2-1}{2x}$$

$$\Rightarrow 1+(\frac{dy}{dx})^2 = 1+(\frac{x^2-1}{2x})^2 = \frac{4x^2+x^4-2x^2+1}{(2x)^2} = \frac{x^4+2x^2+1}{(2x)^2} = \frac{(x^2+1)^2}{(2x)^2}$$

$$\Rightarrow ds = \frac{x^2+1}{2x}dx = (\frac{x}{2}+\frac{1}{2x})\,dx = \frac{1}{2}(x+\frac{1}{x})\,dx$$

1b) $$l = \int_{x_A=1}^{x_B=e} ds = \int_1^e \frac{1}{2}(x+\frac{1}{x})\,dx = \frac{1}{2}[\frac{x^2}{2}+\ln x]_1^e = \frac{1}{2}(\frac{e^4}{2}+1-\frac{1}{2}) = \frac{1}{4}(e^4+1) \approx 2{,}1\,m$$

2) $$I_C = \int_1^e \Phi(x,y)ds = \int_1^e x(\frac{x}{2}+\frac{1}{2x})\,dx = \int_1^e (\frac{x^2}{2}+\frac{1}{2})dx = [\frac{x^3}{6}+\frac{x}{2}]_1^e = \frac{e^3}{6}+\frac{e}{2}-\frac{1}{6}-\frac{1}{2} \approx 4{,}04$$

IC3

$$y(x) = \sqrt{x}(\frac{x}{3}-1) = x^{1/2}(\frac{x}{3}-1) = \frac{x^{3/2}}{3}-x^{1/2} \quad\rightarrow\quad \Phi(x,y) = xy = x(\frac{x^{3/2}}{3}-x^{1/2}) = \frac{x^{5/2}}{3}-x^{3/2}$$

1) $$ds = \sqrt{1+(\frac{dy}{dx})^2}\,dx \quad\rightarrow\quad \frac{dy}{dx} = \frac{3}{2}\frac{x^{1/2}}{3} - \frac{1}{2}x^{-1/2} = \frac{\sqrt{x}}{2}-\frac{1}{2\sqrt{x}}$$

$$\Rightarrow 1+(\frac{dy}{dx})^2 = 1+(\frac{\sqrt{x}}{2}-\frac{1}{2\sqrt{x}})^2 = 1+\frac{x}{4}-2\frac{\sqrt{x}}{2}\frac{1}{2\sqrt{x}}+\frac{1}{4x} = \frac{x}{4}-\frac{1}{2}+\frac{1}{4x} = (\frac{\sqrt{x}}{2}+\frac{1}{2\sqrt{x}})^2$$

$$\Rightarrow ds = (\frac{\sqrt{x}}{2}+\frac{1}{2\sqrt{x}})\,dx = (\frac{x^{1/2}}{2}+\frac{x^{-1/2}}{2})\,dx$$

$$l = \int_{x_A=0}^{x_B=3} ds = \int_0^3 (\frac{x^{1/2}}{2}+\frac{x^{-1/2}}{2})\,dx = [\frac{2}{3}\frac{x^{3/2}}{2}+2\frac{x^{1/2}}{2}]_0^3 = \frac{3^{3/2}}{3}+3^{1/2} = 2\times 3^{1/2} = 2\sqrt{3}$$

2)

$$I_C = \int_0^3 \Phi(x,y)ds = \int_0^3 (\frac{x^{5/2}}{3}-x^{3/2})(\frac{x^{1/2}}{2}+\frac{x^{-1/2}}{2})\,dx = \int_0^3 (\frac{x^3}{6}+\frac{x^2}{6}-\frac{x^2}{2}-\frac{x}{2})\,dx = \int_0^3 (\frac{x^3}{6}-\frac{x^2}{3}-\frac{x}{2})\,dx$$

$$I_C = [\frac{x^4}{24}-\frac{x^3}{9}-\frac{x^2}{4}]_0^3 = (\frac{81}{24}-\frac{27}{9}-\frac{9}{4}) = \frac{81-72-54}{24} = -\frac{45}{24} = -\frac{15}{8}$$

IC4

- $\begin{cases} x = a\cos t \\ y = a\sin t \\ z = ht \end{cases} \quad t\in[0,2\pi] \quad \Rightarrow \begin{cases} dx/dt = -a\sin t \\ dy/dt = a\cos t \\ dz/dt = h \end{cases}$
- $ds = \sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2+(\frac{dz}{dt})^2}\, dt = \sqrt{a^2\sin^2 t+a^2\cos^2 t+h^2}\, dt = \sqrt{a^2+h^2}\, dt$
- $L = \int_0^{2\pi} ds = \int_0^{2\pi} \sqrt{a^2+h^2}\, dt = \sqrt{a^2+h^2}\int_0^{2\pi} dt = 2\pi\sqrt{a^2+h^2}$

IC6

- $\rho = a e^{-\varphi}$, $\varphi\in[0,2\pi]$
- $ds = \sqrt{(d\rho)^2+(\rho\, d\varphi)^2} = \sqrt{(\frac{d\rho}{d\varphi})^2+\rho^2}\ d\varphi = \sqrt{(-a\varphi e^{-\varphi})^2+(a e^{-\varphi})^2}\ d\varphi = \sqrt{2a^2(e^{-\varphi})^2}\ d\varphi$

  $ds = \sqrt{2}\, a e^{-\varphi}\, d\varphi$
- $L = \int_0^{2\pi} ds = \int_0^{2\pi} \sqrt{2}\, a e^{-\varphi}\, d\varphi = a\sqrt{2}[-e^{-\varphi}]_0^{2\pi} = a\sqrt{2}(-e^{-2\pi}-(-e^{-0})) = a\sqrt{2}(1-e^{-2\pi})$

$$\lim_{\varphi\to+\infty} L = a\sqrt{2}(1-e^{-\infty}) = a\sqrt{2}$$

IC9

1) $\Gamma \begin{cases} \rho=R \\ \varphi\in[0,2\pi] \end{cases} \rightarrow \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \end{cases}$, R = cte

- $\lambda(x,y) = C$, C = cte

$Q = \oint_\Gamma C\, dl = C\oint_\Gamma dl = 2\pi RC$, car $\oint_\Gamma dl$ = périmètre du cercle

- $\lambda(x,y) = \frac{Cx^2}{x^2+y^2}$ → coordonnées polaires ⇒ $\lambda(R,\varphi) = \frac{C R^2\cos^2\varphi}{R^2\cos^2\varphi+R^2\sin^2\varphi} = C\cos^2\varphi$

$dl = \sqrt{(d\rho)^2+(\rho\, d\varphi)^2} = \sqrt{(dR)^2+(Rd\varphi)^2} = R\, d\varphi$

$Q = \oint_0^{2\pi} C\cos^2(\varphi) R\, d\varphi = RC\oint_0^{2\pi} \frac{1+\cos(2\varphi)}{2} d\varphi = \frac{RC}{2}[\varphi+\sin(2\varphi)]_0^{2\pi} = \pi RC$

2) $\Sigma \begin{cases} \rho\in[0,R] \\ \varphi\in[0,2\pi] \end{cases} \rightarrow \begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases}$

- $\sigma(x,y) = C$, C = cte

$Q = \oiint_\Sigma C\, dS = C\oiint_\Sigma dS = \pi R^2 C$, car $\oiint_\Sigma dS$ = aire du disque

- $\sigma(x,y) = C\sqrt{x^2+y^2}$ → coordonnées polaires ⇒ $\sigma(\rho,\varphi) = C\rho$

$dS = \rho\, d\rho\, d\varphi$

$Q = \int_0^{2\pi} d\varphi \int_0^{R} C\rho^2 d\rho = 2\pi C\frac{R^3}{3}$

3) E $\begin{cases} r \in [0, R] \\ \theta \in [0, \pi] \\ \varphi \in [0, 2\pi] \end{cases} \rightarrow \begin{cases} x = r \sin\theta \cos\varphi \\ y = r \sin\theta \sin\varphi \\ z = r \cos\theta \end{cases}$

- $\rho(x, y, z) = C$, C = cte

$Q = \oiiint_E C\, dV = C \oiiint_E dV = \frac{4}{3}\pi R^3 C$, car $\oiint_\Sigma dS$ = volume de la sphère

- $\rho(x, y, z) = C\sqrt{x^2 + y^2 + z^2}$ $\rightarrow$ coordonnées sphérq $\Rightarrow$ $\rho(r, \theta, \varphi) = Cr$

$dV = r^2 \sin\theta\, dr\, d\theta\, d\varphi$

$Q = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta \int_0^{R} C r^3 dr = 2\pi 2C \frac{R^4}{4} = \pi C R^4$

Notes :
cte : constante(s)
-q : -ique(s)