Outils Mathématiques – Calcul Intégral

Intégrales Doubles

ID1

$$I = \iint_D xy\,dxdy = \int_0^1 x\left(\int_0^{1-x} y\,dy\right)dx = \int_0^1 x\left[\frac{y^2}{2}\right]_0^{1-x} dx = \int_0^1 x\frac{(1-x)^2}{2}dx = \frac{1}{2}\int_0^1 (x^3-2x^2+x)\,dx$$

$$I = \frac{1}{2}\left[\frac{x^4}{4}-\frac{2}{3}x^3+\frac{x^2}{2}\right]_0^1 = \frac{1}{2}\left[\frac{1}{4}-\frac{2}{3}+\frac{1}{2}\right] = \frac{1}{2}\left[\frac{3-8+6}{12}\right] = \frac{1}{24}$$

ID3

1) $$S_D = \iint_D 1\,dxdy = \int_0^1 \left(\int_{x^2}^{\sqrt{x}} dy\right)dx = \int_0^1 [y]_{x^2}^{\sqrt{x}} dx = \int_0^1 (\sqrt{x}-x^2)\,dx = \int_0^1 (x^{1/2}-x^2)\,dx = \left[\frac{2}{3}x^{3/2}-\frac{x^3}{3}\right]_0^1$$

$$S_D = \frac{2}{3}-\frac{1}{3} = \frac{1}{3}$$

2) $$I = \iint_D xy\,dxdy = \int_0^1 x\left(\int_{x^2}^{\sqrt{x}} y\,dy\right)dx = \int_0^1 x\left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx = \int_0^1 x\left(\frac{x}{2}-\frac{x^4}{2}\right)dx = \int_0^1 \left(\frac{x^2}{2}-\frac{x^5}{2}\right)dx$$

$$I = \frac{1}{2}\left[\frac{x^3}{3}-\frac{x^6}{6}\right]_0^1 = \frac{1}{2}\left(\frac{1}{3}-\frac{1}{6}\right) = \frac{1}{12}$$

ID4

Surface de l'ellipse ?

$y_-(x) = 0$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \iff y^2 = b^2\left(1-\frac{x^2}{a^2}\right)$$

$y_+(x) = ?$ → $$\iff y = \pm b\sqrt{1-\frac{x^2}{a^2}}$$

$$\Rightarrow y_+(x) = b\sqrt{1-\frac{x^2}{a^2}}$$

$S = 4S_{D_1}$, avec D_1 le quart supérieur droit de l'ellipse (tel que $x \geqslant 0$ et $y \geqslant 0$)

Donc $$S = 4\left(\iint_{D_1} dxdy\right) = 4\left[\int_0^a \left(\int_0^{b\sqrt{1-x^2/a^2}} dy\right)dx\right] = 4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}dx = 4b\int_0^a \sqrt{1-\frac{x^2}{a^2}}dx$$

Les calculs mènent à $S = 4b\frac{\pi a}{4} = \pi ab$

ID5

Rappel : équation d'un cercle → $x^2+y^2 = R^2$, aire d'un cercle = πR^2

$$x_G = \frac{1}{S}\iint_D x\,dxdy = \frac{2}{\pi R^2}\int_{-R}^{R} x\left(\int_0^{\sqrt{R^2-x^2}} dy\right)dx = \frac{2}{\pi R^2}\int_{-R}^{R} x\sqrt{R^2-x^2}\,dx = \left(-\frac{1}{2}\frac{2}{3}\right)\frac{2}{\pi R^2}\int_{-R}^{R}\left(-2\frac{3}{2}\right)x(R^2-x^2)^{1/2}dx$$

$$x_G = -\frac{2}{3\pi R^2}\left[(R^2-x^2)^{3/2}\right]_{-R}^{R} = 0$$

$$y_G = \frac{1}{S}\iint_D y dx dy = \frac{2}{\pi R^2}\int_{-R}^{R}\left(\int_0^{\sqrt{R^2-x^2}} y dy\right)dx = \frac{2}{\pi R^2}\int_{-R}^{R}\left[\frac{y^2}{2}\right]_0^{\sqrt{R^2-x^2}} dx = \frac{2}{\pi R^2}\int_{-R}^{R}\frac{R^2-x^2}{2}dx$$

$$y_G = \frac{1}{\pi R^2}\left[xR^2 - \frac{x^3}{3}\right]_{-R}^{R} = \frac{1}{\pi R^2}\left[R^3 - \frac{R^3}{3} - \left(-R^3 - \frac{-R^3}{3}\right)\right] = \frac{1}{\pi R^2}\left(2R^3 - \frac{2R^3}{3}\right) = \frac{1}{\pi R^2}\left(\frac{4R^3}{3}\right) = \frac{4R}{3\pi}$$

En polaire

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases} \qquad ds = \rho\, d\rho\, d\varphi \qquad D' \begin{cases} \rho\in[0,R] \\ \varphi\in[0,\pi] \end{cases}$$

$$y_G = \frac{2}{R^2\pi}\int_0^{\pi}\left(\int_0^{R}\rho\sin\varphi\,\rho\, d\rho\right)d\varphi = \frac{2}{R^2\pi}\int_0^{\pi}\sin\varphi\left[\frac{\rho^3}{3}\right]_0^{R}d\varphi = \frac{2}{R^2\pi}\int_0^{\pi}\sin\varphi\frac{R^3}{3}d\varphi = \frac{2R}{3\pi}[-\cos\varphi]_0^{\pi}$$

$$y_G = \frac{4R}{3\pi}$$

ID6

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases} \qquad f(x,y) = \mu(x^2+y^2) \;\rightarrow\; g(\rho,\varphi) = \mu\rho^2 \qquad ds = \rho\, d\rho\, d\varphi \qquad D' \begin{cases} \rho\in[0,R] \\ \varphi\in[0,2\pi] \end{cases}$$

$$I_0 = \int_0^{2\pi}\left(\int_0^{R}\mu\rho^2\rho\, d\rho\right)d\varphi = \mu\int_0^{2\pi}d\varphi\int_0^{R}\rho^3 d\rho = \mu 2\pi\left[\frac{\rho^4}{4}\right]_0^{R} = \mu\pi\frac{R^4}{2}$$

Rmq : masse → $M = \mu\times S = \mu\pi R^2$, S la surface de la plaque

$$I_0 = \frac{MR^2}{2}$$

ID9

1) Les coordonnées polaires (ρ,φ)

2) $dS = \rho\, d\rho\, d\varphi$ et $dm = \sigma(\rho) dS$

3) $m = \iint_\Sigma dm = \iint_\Sigma \sigma(\rho)\rho\, d\rho\, d\varphi = \iint_\Sigma a(R^2-\rho^2)\rho\, d\rho\, d\varphi$, avec $\Sigma \begin{cases} \rho\in[0,R] \\ \varphi\in[0,2\pi] \end{cases}$

Donc $m = a\int_0^{2\pi}d\varphi\int_0^{R}R^2\rho - \rho^3 d\rho = a2\pi\left[\frac{R^2\rho^2}{2} - \frac{\rho^4}{4}\right]_0^{R} = 2\pi a\left(\frac{R^4}{2} - \frac{R^4}{4}\right) = \frac{\pi a R^4}{4}$

4)

- $R^4 \;\alpha\; m^4$, ici m = mètres
- $a = \frac{\sigma(\rho)}{(R^2-\rho^2)} \;\alpha\; \frac{kg\cdot m^{-2}}{m^2}$ donc $a \;\alpha\; kg\cdot m^{-4}$
- Finalement, on a bien $m = \frac{\pi a R^4}{4} \;\alpha\; kg\cdot m^{-4}\cdot m^4 = kg$