Outils Mathématiques – Systèmes de coordonnées

SC1

1) cartésiennes : $\overrightarrow{OM} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$
cylindrq : $\overrightarrow{OM} = \rho\vec{e}_\rho + z\vec{e}_z$
sphérq : $\overrightarrow{OM} = r\vec{e}_r$

2)

Cylindrq : $(\rho,\varphi,z)\begin{Bmatrix} \vec{e}_\rho = \cos\varphi\,\vec{e}_x + \sin\varphi\,\vec{e}_y \\ \vec{e}_\varphi = -\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y \\ \vec{e}_z \end{Bmatrix}$ Sphérq : $(r,\theta,\varphi)\begin{Bmatrix} \vec{e}_r = \sin\theta\cos\varphi\,\vec{e}_x + \sin\theta\sin\varphi\,\vec{e}_y + \cos\theta\,\vec{e}_z \\ \vec{e}_\theta = \cos\theta\cos\varphi\,\vec{e}_x + \cos\theta\sin\varphi\,\vec{e}_y - \sin\theta\,\vec{e}_z \\ \vec{e}_\varphi = -\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y \end{Bmatrix}$

Cylindrq :

- $\dot{\vec{e}}_\rho = \frac{d}{dt}(\cos\varphi\,\vec{e}_x + \sin\varphi\,\vec{e}_y) = \frac{d}{dt}(\cos\varphi)\vec{e}_x + \frac{d}{dt}(\sin\varphi)\vec{e}_y = -\frac{d\varphi}{dt}\sin\varphi\,\vec{e}_x + \frac{d\varphi}{dt}\cos\varphi\,\vec{e}_y$
  $\dot{\vec{e}}_\rho = -\dot{\varphi}\sin\varphi\,\vec{e}_x + \dot{\varphi}\cos\varphi\,\vec{e}_y = \dot{\varphi}(-\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y)$
  $\dot{\vec{e}}_\rho = \dot{\varphi}\vec{e}_\varphi$
- $\dot{\vec{e}}_\varphi = \frac{d}{dt}(-\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y) = -\dot{\varphi}\cos\varphi\,\vec{e}_x - \dot{\varphi}\sin\varphi\,\vec{e}_y = -\dot{\varphi}(\cos\varphi\,\vec{e}_x + \sin\varphi\,\vec{e}_y)$
  $\dot{\vec{e}}_\varphi = -\dot{\varphi}\vec{e}_\rho$
- $\dot{\vec{e}}_z = \vec{O}$

Sphérq :

- $\dot{\vec{e}}_r = \frac{d\vec{e}_r}{dt} = (\dot{\theta}\cos\theta\cos\varphi - \dot{\varphi}\sin\theta\sin\varphi)\vec{e}_x + (\dot{\theta}\cos\theta\sin\varphi + \dot{\varphi}\sin\theta\cos\varphi)\vec{e}_y - \dot{\theta}\sin\theta\,\vec{e}_z$
  $\dot{\vec{e}}_r = \dot{\theta}(\cos\theta\cos\varphi\,\vec{e}_x + \cos\theta\sin\varphi\,\vec{e}_y - \sin\theta\,\vec{e}_z) + \dot{\varphi}(-\sin\theta\sin\varphi\,\vec{e}_x + \sin\theta\cos\varphi\,\vec{e}_y)$
  $\dot{\vec{e}}_r = \dot{\theta}\vec{e}_\theta + \dot{\varphi}\sin\theta\,\vec{e}_\varphi$
- $\dot{\vec{e}}_\theta = \frac{d\vec{e}_\theta}{dt} = (-\dot{\theta}\sin\theta\cos\varphi - \dot{\varphi}\cos\theta\sin\varphi)\vec{e}_x + (-\dot{\theta}\sin\theta\sin\varphi + \dot{\varphi}\cos\theta\cos\varphi)\vec{e}_y - \dot{\theta}\cos\theta\,\vec{e}_z$
  $\dot{\vec{e}}_\theta = -\dot{\theta}(\sin\theta\cos\varphi\,\vec{e}_x + \sin\theta\sin\varphi\,\vec{e}_y + \cos\theta\,\vec{e}_z) + \dot{\varphi}\cos\theta(-\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y)$
  $\dot{\vec{e}}_\theta = -\dot{\theta}\vec{e}_r + \dot{\varphi}\cos\theta\,\vec{e}_\varphi$
- $\dot{\vec{e}}_\varphi = -\dot{\varphi}\cos\varphi\,\vec{e}_x - \dot{\varphi}\sin\varphi\,\vec{e}_y = -\dot{\varphi}(\cos\varphi\,\vec{e}_x + \sin\varphi\,\vec{e}_y)$
  On voit que $\sin\theta\,\vec{e}_r + \cos\theta\,\vec{e}_\theta = \cos\varphi\,\vec{e}_x + \sin\varphi\,\vec{e}_y$
  $\dot{\vec{e}}_\varphi = -\dot{\varphi}(\sin\theta\,\vec{e}_r + \cos\theta\,\vec{e}_\theta)$

3)

vecteur vitesse : $\vec{v} = \frac{d\overrightarrow{OM}}{dt}$

- cartésiennes : $\vec{v} = \dot{x}\vec{e}_x + \dot{y}\vec{e}_y + \dot{z}\vec{e}_z$
- cylindrq : $\vec{v} = \dot{\rho}\vec{e}_\rho + \rho\dot{\vec{e}}_\rho + \dot{z}\vec{e}_z(+z\dot{\vec{e}}_z = \vec{O}) = \dot{\rho}\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi + \dot{z}\vec{e}_z$
- sphérq : $\vec{v} = \dot{r}\vec{e}_r + r\dot{\vec{e}}_r = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + r\dot{\varphi}\sin\theta\,\vec{e}_\varphi$

vecteur accélérat° : $\vec{\gamma} = \frac{d\vec{v}}{dt}$

- cartésiennes : $\vec{\gamma} = \ddot{x}\vec{e}_x + \ddot{y}\vec{e}_y + \ddot{z}\vec{e}_z$
- cylindrq : $\vec{\gamma} = \ddot{\rho}\vec{e}_\rho + \dot{\rho}\dot{\vec{e}}_\rho + \dot{\rho}\dot{\varphi}\vec{e}_\varphi + \rho\ddot{\varphi}\vec{e}_\varphi + \rho\dot{\varphi}\dot{\vec{e}}_\varphi + \ddot{z}\vec{e}_z(+\dot{z}\dot{\vec{e}}_z = 0)$
  $\vec{\gamma} = \ddot{\rho}\vec{e}_\rho + \dot{\rho}\dot{\varphi}\vec{e}_\varphi + \dot{\rho}\dot{\varphi}\vec{e}_\varphi + \rho\ddot{\varphi}\vec{e}_\varphi - \rho\dot{\varphi}\dot{\varphi}\vec{e}_\rho + \ddot{z}\vec{e}_z$
  $\vec{\gamma} = (\ddot{\rho} - \rho\dot{\varphi}^2)\vec{e}_\rho + (2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi})\vec{e}_\varphi + \ddot{z}\vec{e}_z$

- sphérq : $\vec{\gamma} = \ddot{r}\,\vec{e}_r + \dot{r}\,\dot{\vec{e}}_r + \dot{r}\,\dot{\theta}\,\vec{e}_\theta + r\,\ddot{\theta}\,\vec{e}_\theta + r\,\dot{\theta}\,\dot{\vec{e}}_\theta + \dot{r}\,\dot{\varphi}\sin\theta\,\vec{e}_\varphi + r\,\ddot{\varphi}\sin\theta\,\vec{e}_\varphi + r\,\dot{\varphi}\,\dot{\theta}\cos\theta\,\vec{e}_\varphi + r\,\dot{\varphi}\sin\theta\,\dot{\vec{e}}_\varphi$

$$\vec{\gamma} = \ddot{r}\,\vec{e}_r + \dot{r}(\dot{\theta}\,\vec{e}_\theta + \dot{\varphi}\sin\theta\,\vec{e}_\varphi) + \dot{r}\,\dot{\theta}\,\vec{e}_\theta + r\,\ddot{\theta}\,\vec{e}_\theta + r\,\dot{\theta}(-\dot{\theta}\,\vec{e}_r + \dot{\varphi}\cos\theta\,\vec{e}_\varphi) + \dot{r}\,\dot{\varphi}\sin\theta\,\vec{e}_\varphi$$
$$+ r\,\ddot{\varphi}\sin\theta\,\vec{e}_\varphi + r\,\dot{\varphi}\,\dot{\theta}\cos\theta\,\vec{e}_\varphi + r\,\dot{\varphi}\sin\theta(-\dot{\varphi}\sin\theta\,\vec{e}_r - \dot{\varphi}\cos\theta\,\vec{e}_\theta)$$
$$\vec{\gamma} = (\ddot{r} - r\,\dot{\theta}^2 - r\,\dot{\varphi}^2\sin^2\theta)\,\vec{e}_r + (2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta} - r\,\dot{\varphi}^2\sin\theta\cos\theta)\,\vec{e}_\theta + (2\,\dot{r}\,\dot{\varphi}\sin\theta + 2\,r\,\dot{\theta}\,\dot{\varphi}\cos\theta + r\,\ddot{\varphi}\sin\theta)\,\vec{e}_\varphi$$

4) $\begin{cases} \rho(t) = R \\ \varphi(t) = \omega t \end{cases}$ $\quad \overrightarrow{OM}(t) = \rho\,\vec{e}_\rho = R\,\vec{e}_\rho$

$\vec{v}(t) = R\,\dot{\vec{e}}_\rho = R\,\dot{\varphi}\,\vec{e}_\varphi$ or $\dot{\varphi} = \omega$

donc $\vec{v} = R\,\omega\,\vec{e}_\varphi$ donc la vitesse est tangentielle à tout moment

$\vec{\gamma}(t) = R\,\omega\,\dot{\vec{e}}_\varphi = -R\,\omega\,\dot{\varphi}\,\vec{e}_\rho$ donc $\vec{\gamma}(t) = -R\,\omega^2\,\vec{e}_\rho$ → accélérat° centrale

SC2

1) rayon ρ ; angle polaire φ ; distance z

2) $\begin{Bmatrix} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z \end{Bmatrix}$ $\quad$ 3) $\vec{e}_u = \dfrac{\partial\overrightarrow{OM}/\partial u}{\|\partial\overrightarrow{OM}/\partial u\|}$, $\begin{Bmatrix} \vec{e}_\rho = \cos\varphi\,\vec{e}_x + \sin\varphi\,\vec{e}_y \\ \vec{e}_\varphi = -\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y \\ \vec{e}_z \end{Bmatrix}$

4) $\overrightarrow{OM} = \rho\,\vec{e}_\rho + z\,\vec{e}_z$

5) $\begin{Bmatrix} \rho(t) = R \\ \varphi(t) = \omega t \\ z(t) = \lambda\omega t \end{Bmatrix}$, avec R, ω, λ constantes

5a) $\vec{v} = \dot{\rho}\,\vec{e}_\rho + \rho\,\dot{\vec{e}}_\rho + \dot{z}\,\vec{e}_z = \vec{O} + \rho\,\dot{\varphi}\,\vec{e}_\varphi + \dot{z}\,\vec{e}_z = R\,\omega\,\vec{e}_\varphi + \lambda\,\omega\,\vec{e}_z$

5b) $\vec{\gamma} = \dfrac{d}{dt}(R\,\omega\,\vec{e}_\varphi + \lambda\,\omega\,\vec{e}_z) = R\,\omega\,\dot{\vec{e}}_\varphi = -R\,\omega\,\dot{\varphi}\,\vec{e}_\rho = -R\,\omega^2\,\vec{e}_\rho$

5c) 1e méthode : $\tan\alpha = \dfrac{v_z}{v_\varphi} = \dfrac{\lambda\,\omega}{R\,\omega} = \dfrac{\lambda}{R} \Rightarrow \alpha = \tan^{-1}(\dfrac{\lambda}{R})$

2e méthode : $\vec{v}\cdot\vec{e}_\varphi = \|\vec{v}\|\|\vec{e}_\varphi\|\cos(\vec{v},\vec{e}_\varphi) = \|\vec{v}\|\cos(\vec{v},\vec{e}_\varphi)$

donc $\cos\alpha = \dfrac{\vec{v}\cdot\vec{e}_\varphi}{\|\vec{v}\|} = \dfrac{R\,\omega}{\sqrt{R^2\omega^2 + \lambda^2\omega^2}} = \dfrac{R}{\sqrt{R^2 + \lambda^2}} = \dfrac{1}{\sqrt{1 + (\lambda/R)^2}}$

donc $\alpha = \cos^{-1}(\dfrac{1}{\sqrt{1 + (\lambda/R)^2}})$

Notes :
q : -ique(s)
° : -ion(s)