# Outils Mathématiques – Calcul Différentiel

**CD1**

1) $$\frac{\partial f}{\partial x}=\frac{1}{x-y}\cos(x^2+y^2)+\ln(|x-y|)(-2x\sin(x^2+y^2))$$
$$\frac{\partial f}{\partial y}=-\frac{1}{x-y}\cos(x^2+y^2)+\ln(|x-y|)(-2y\sin(x^2+y^2))$$
Finalement :
$$df=[\frac{\cos(x^2+y^2)}{x-y}-\ln(|x-y|)2x\sin(x^2+y^2)]dx+[\frac{-\cos(x^2+y^2)}{x-y}-\ln(|x-y|)2y\sin(x^2+y^2)]dy$$

2) Soit $f(x,y)=x\cos y+y\exp(x)$

$$\frac{\partial f}{\partial x}=\cos(y)+y\,e^x \qquad \frac{\partial^2 f}{\partial y\partial x}=\frac{\partial f}{\partial y}(\frac{\partial f}{\partial x})=\frac{\partial f}{\partial y}(\cos(y)+y\,e^x)=-\sin(y)+e^x$$
$$\frac{\partial f}{\partial y}=-x\sin(y)+e^x \qquad \frac{\partial^2 f}{\partial x\partial y}=\frac{\partial f}{\partial x}(-x\sin(y)+e^x)=-\sin(y)+e^x$$
$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial f}{\partial x}(\cos(y)+y\,e^x)=y\,e^x \qquad \frac{\partial^2 f}{\partial y^2}=\frac{\partial f}{\partial y}(-x\sin(y)+e^x)=-x\cos(y)$$

Soit $f(x,y)=x^2\exp(xy)$

$$\frac{\partial f}{\partial x}=2x\,e^{xy}+x^2y\,e^{xy} \qquad \frac{\partial^2 f}{\partial y\partial x}=2x^2e^{xy}+x^2e^{xy}+x^3y\,e^{xy}=3x^2e^{xy}+x^3y\,e^{xy}$$
$$\frac{\partial f}{\partial y}=x^2x\,e^{xy}=x^3e^{xy} \qquad \frac{\partial^2 f}{\partial x\partial y}=3x^2e^{xy}+x^3y\,e^{xy}$$
$$\frac{\partial^2 f}{\partial y^2}=x^3x\,e^{xy}=x^4e^{xy} \qquad \frac{\partial^2 f}{\partial x^2}=2e^{xy}+2xy\,e^{xy}+y\,2x\,e^{xy}+yx^2y\,e^{xy}=2e^{xy}+4xy\,e^{xy}+x^2y^2e^{xy}$$

**CD3**

1) Soit $f(x,y)=\sin(xy)$

$$\frac{\partial f}{\partial x}=y\cos xy \qquad \frac{\partial f}{\partial y}=x\cos xy$$

2) $$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=(\frac{\partial f}{\partial u})\times1+(\frac{\partial f}{\partial v})\times1=\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}$$
$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=(\frac{\partial f}{\partial u})\times1+(\frac{\partial f}{\partial v})\times(-1)=\frac{\partial f}{\partial u}-\frac{\partial f}{\partial v}$$

3) Exprimons y puis y en fonction de u et v :

$$\begin{Bmatrix}u=x+y\\v=x-y\end{Bmatrix}\Leftrightarrow\begin{Bmatrix}u+v=x+y+x-y\\u-v=x+y-x+y\end{Bmatrix}\Leftrightarrow\begin{Bmatrix}2x=u+v\\2y=u-v\end{Bmatrix}\Leftrightarrow\begin{Bmatrix}x=\frac{u+v}{2}\\y=\frac{u-v}{2}\end{Bmatrix}$$

$$f(u,v)=\sin(\frac{u+v}{2}\frac{u-v}{2})=\sin(\frac{u^2-v^2}{4})$$

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}=\frac{2u}{4}\cos(\frac{u^2-v^2}{4})+(\frac{-2v}{4}\cos(\frac{u^2-v^2}{4}))=\frac{u-v}{2}\cos(\frac{u^2-v^2}{4})=y\cos xy$$
$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}-\frac{\partial f}{\partial v}=\frac{2u}{4}\cos(\frac{u^2-v^2}{4})-(\frac{-2v}{4}\cos(\frac{u^2-v^2}{4}))=\frac{u+v}{2}\cos(\frac{u^2-v^2}{4})=x\cos xy$$

**CD4**

Loi de Snell : $n_1\sin i_1=n_2\sin i_2\Leftrightarrow i_2=\arcsin(\frac{n_1}{n_2}\sin i_1)$

1) Calcul direct : $di_2=\frac{n_1}{n_2}\frac{\cos i_1}{\sqrt{1-(\frac{n_1}{n_2}\sin i_1)^2}}di_1=\frac{n_1}{n_2}\frac{\cos i_1}{\sqrt{(\cos i_2)^2}}di_1=\frac{n_1\cos i_1}{n_2\cos i_2}di_1$

2) Calcul indirect : $n_1\sin i_1=n_2\sin i_2\Leftrightarrow d(n_1\sin i_1)=d(n_2\sin i_2)$

$$\Leftrightarrow n_1 d(\sin i_1)=n_2 d(\sin i_2)$$
$$\Leftrightarrow n_1\cos(i_1)di_1=n_2\cos(i_2)di_2$$
$$\Leftrightarrow di_2=\frac{n_1\cos i_1}{n_2\cos i_2}di_1$$

donc $di_2\approx 0{,}07712°$

or $\Delta i_1=0{,}1°$ ce qui induit une variat° $\Delta i_2\approx d i_2\approx 0{,}07712°$

3) $\sin i_2=\frac{n_1}{n_2}\sin i_1\Rightarrow i_2=37{,}639°$

Valeur exacte de $\Delta i_2$

$$n_1\sin(i_1+\Delta i_1)=n_2\sin(i_2+\Delta i_2)\Rightarrow\sin(i_2+\Delta i_2)=\frac{n_1}{n_2}\sin(i_1+\Delta i_1)$$
$$\Rightarrow i_2+\Delta i_2=37{,}716°$$
$$\Rightarrow\Delta i_2=0{,}0770°$$

**CD5**

1) $\frac{\partial P}{\partial T}=\frac{r}{V-b}e^{-\frac{a}{rTV}}+\frac{rT}{V-b}(-\frac{a}{rV})(-\frac{1}{T^2})e^{-\frac{a}{rTV}}=\frac{r}{V-b}e^{-\frac{a}{rTV}}(1+\frac{a}{rVT})$

$$\frac{\partial P}{\partial V}=\frac{-rT}{(V-b)^2}e^{-\frac{a}{rTV}}+\frac{rT}{V-b}(-\frac{a}{rT})(-\frac{1}{V^2})e^{-\frac{a}{rTV}}=\frac{rT}{V-b}e^{-\frac{a}{rTV}}(\frac{a}{rTV^2}-\frac{1}{V-b})$$
$$dP=[\frac{r}{V-b}e^{-\frac{a}{rTV}}(1+\frac{a}{rVT})]dT+[\frac{rT}{V-b}e^{-\frac{a}{rTV}}(\frac{a}{rTV^2}-\frac{1}{V-b})]dV$$

2a) $\beta=\frac{1}{P}\frac{\partial P}{\partial T}=\frac{V-b}{rT}\frac{1}{e^{-\frac{a}{rTV}}}\frac{r}{V-b}e^{-\frac{a}{rTV}}(1+\frac{a}{rVT})=\frac{1}{T}(1+\frac{a}{rVT})$

b) On ne connaît pas $\frac{\partial V}{\partial T}$ (ni $\frac{\partial V}{\partial P}$ pour la qs 2c) )

$dV=\frac{\partial V}{\partial P}dP+\frac{\partial V}{\partial T}dT$ (1) et $dP=\frac{\partial P}{\partial T}dT+\frac{\partial P}{\partial V}dV\Leftrightarrow\frac{\partial P}{\partial V}dV=dP-\frac{\partial P}{\partial T}dT$

$$\Leftrightarrow dV=\frac{1}{(\partial P/\partial V)}dP-\frac{(\partial P/\partial T)}{(\partial P/\partial V)}dT \quad (2)$$

Donc de (1) et (2), il résulte que : $\begin{cases} \dfrac{\partial V}{\partial P} = \dfrac{1}{\partial P / \partial V} \\ \dfrac{\partial V}{\partial T} = \dfrac{-\partial P / \partial T}{\partial P / \partial V} \end{cases}$

Donc $\alpha = \frac{1}{V}\frac{\partial V}{\partial T} = -\frac{1}{V}\frac{\frac{r}{V-b}e^{-\frac{a}{rTV}}(1+\frac{a}{rTV})}{\frac{rT}{V-b}e^{-\frac{a}{rTV}}(\frac{a}{rTV^2}-\frac{1}{V-b})} = -\frac{1}{V}\frac{\frac{rTV+a}{rTV}}{T\frac{a(V-b)-rTV^2}{rTV^2(V-b)}} = -\frac{1}{V}\frac{1}{T}\frac{rTV+a}{rTV}\frac{rTV^2(V-b)}{a(V-b)-rTV^2}$

$$\alpha = -\frac{(rTV+a)(V-b)}{aT(V-b)-rT^2V^2}$$

c) $\chi_T = -\frac{1}{V}\frac{\partial V}{\partial P} = -\frac{1}{V}\frac{1}{\frac{rT}{V-b}e^{-\frac{a}{rTV}}(\frac{a}{rTV^2}-\frac{1}{V-b})} = -\frac{1}{V}\frac{V-b}{rT}e^{\frac{a}{rTV}}\frac{rTV^2(V-b)}{a(V-b)-rTV^2}$

$$\chi_T = -\frac{V(V-b)^2}{a(V-b)-rTV^2}e^{\frac{a}{rTV}}$$

**CD8**

$df = (3y^2 - y^3 - x^2)dx + 3xy(2-y)dy \quad \Leftrightarrow \begin{cases} \dfrac{\partial f}{\partial x} = 3y^2 - y^3 - x^2 & (1) \\ \dfrac{\partial f}{\partial y} = 3xy(2-y) & (2) \end{cases}$

(1) : $f(x,y) = 3xy^2 - xy^3 - \frac{x^3}{3} + G(y)$

$\frac{\partial f}{\partial y} = 6xy - 3xy^2 + \frac{dG}{dy} = 3xy(2-y) + \frac{dG}{dy}$ , donc d'après (2), $\frac{dG}{dy} = 0 \Leftrightarrow G(y) = C$

or f(1,0) = 1

donc $3\times1\times0^2 - 1\times0^3 - \frac{1^3}{3} + C = 1 \quad \Leftrightarrow -\frac{1}{3} + C = 1 \Leftrightarrow C = \frac{4}{3}$

Finalement : $f(x,y) = 3xy^2 - xy^3 - \frac{x^3}{3} + \frac{4}{3}$

**CD9**

> $\delta Q = C_V dT + \frac{RT}{V}dV = A\,dT + B\,dV$

$\frac{\partial A}{\partial V} = 0$ et $\frac{\partial B}{\partial T} = \frac{R}{V} \neq \frac{\partial A}{\partial V}$ donc NON

> $\frac{\delta Q}{T} = \frac{C_V}{T}dT + \frac{R}{V}dV = A\,dT + B\,dV$

$\frac{\partial A}{\partial V} = 0$ et $\frac{\partial B}{\partial T} = 0 = \frac{\partial A}{\partial V}$ donc OUI $\quad \exists f / \frac{\delta Q}{T} = df \Rightarrow \begin{cases} A(T,V) = \dfrac{\partial f}{\partial T} = \dfrac{C_V}{T} & (1) \\ B(T,V) = \dfrac{\partial f}{\partial V} = \dfrac{R}{V} & (2) \end{cases}$

(1) : $f(T,V)=C_V\ln(T)+G(V)$

$\frac{\partial f}{\partial V}=\frac{dG}{dV}$ , donc d'après (2), $\frac{dG}{dV}=\frac{R}{V}\Leftrightarrow G(V)=R\ln(V)+C$

Finalement : $f(T,V)=C_V\ln(T)+R\ln(T)+C$

**CD11**

$\delta W=F(\sin(\varphi)d\rho+\rho\cos(\varphi)d\varphi)=Fsin(\varphi)d\rho+F\rho\cos(\varphi)d\varphi=A\,d\rho+B\,d\varphi$

$\frac{\partial A}{\partial\varphi}=Fcos\varphi$ et $\frac{\partial B}{\partial\rho}=Fcos\varphi=\frac{\partial A}{\partial\varphi}$ donc OUI, $\exists f/\delta W=df\Rightarrow\begin{cases}A(\rho,\varphi)=\frac{\partial f}{\partial\rho}=Fsin(\varphi) & (1)\\ B(\rho,\varphi)=\frac{\partial f}{\partial\varphi}=F\rho\cos(\varphi) & (2)\end{cases}$

(1) : $f(\rho,\varphi)=F\rho\sin(\varphi)+G(\varphi)$

$\frac{\partial f}{\partial\varphi}=F\rho\cos(\varphi)+\frac{dG}{d\varphi}$ , donc d'après (2), $\frac{dG}{d\varphi}=0\Leftrightarrow G(\varphi)=C$

Finalement : $f(\rho,\varphi)=F\rho\sin(\varphi)+C$

Or f = - $E_p$ , donc $E_p(\rho,\varphi)=-F\rho\sin(\varphi)+C'$

**CD13**

1) $\frac{1}{R}=\frac{1}{2200}+\frac{1}{120}=\frac{29}{3300}\Rightarrow R=\frac{3300}{29}\,\Omega\approx 113{,}7931034\,\Omega$

2) Calcul direct

$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}\Leftrightarrow R=\frac{R_1R_2}{R_1+R_2}$ ; $\Delta R_1=2200\times\frac{15}{100}=330\,\Omega$ ; $\Delta R_1=120\times\frac{15}{100}=18\,\Omega$

$\frac{\partial R}{\partial R_1}=\frac{R_2(R_1+R_2)+R_1R_2}{(R_1+R_2)^2}=\frac{R_2^2}{(R_1+R_2)^2}$ et $\frac{\partial R}{\partial R_2}=\frac{R_1^2}{(R_1+R_2)^2}$

$\Delta R=\frac{R_2^2}{(R_1+R_2)^2}\Delta R_1+\frac{R_1^2}{(R_1+R_2)^2}\Delta R_2\approx 17{,}06896\,\Omega$

Finalement : $\Delta R\leqslant 20\,\Omega$

donc $R=(110\pm 20)\Omega$

Rmq : il est acceptable d'écrire $\Delta R=17\,\Omega$

$R=(114\pm 17)\Omega$

Prochain TD :

- CD13, 2), avec les différentielles logarithmiques
- CD14
- CD15
- CD16